GJ

The University of Calgary
Department of Chemical & Petroleum Engineering

ENCH 501: Transport Phenomena Quiz #5 **November 8, 2011**

Time Allowed: 45 mins. **Name:**

(a) (10 pts) Show that the Integral momentum equation of the boundary layer is given by:

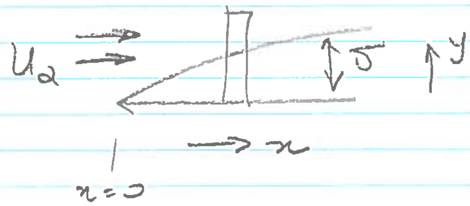
$$\mu \frac{du}{dy}\Big|_{y=0} = \frac{d}{dx}\left(\int_0^\delta \rho(U_\infty - u)u\,dy\right)$$

where δ is the boundary layer thickness. Derive all important steps and state all assumptions.

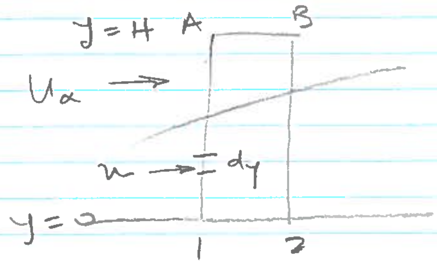
(b) (Bonus: 2 pts) If you assume that $u = a + by + cy^2$, based on the boundary conditions:
$y=0$, $u = 0$; $y = \delta$, $u = U_\infty$; $y = \delta$, $du/dy = 0$,
obtain an expression for $\delta(x)$.

ENCH 501 Quiz #5 Solution Nov. 8, 2011

(a)



Choose differential element as shown — x to $x+dx$ and $y=0$ to $y=H$.



— Material balance

input A1 $= \int_0^H \rho u \, dy$

output B2 $= \int_0^H \rho u \, dy + \frac{d}{dx}\left[\int_0^H \rho u \, dy\right] dx$

Output at AB is the difference, or

$$- \frac{d}{dx}\left[\int_0^H \rho u \, dy\right] dx$$

— Momentum or Force Balance — Newton's law

Net Force = Rate of change of Momentum

Momentum Rate in A1 $= \int_0^H \rho u^2 dy$

out B2 $= \int_0^H \rho u^2 dy + \frac{d}{dx}\left[\int_0^H \rho u^2 dy\right] dx$

out AB $= - U_\alpha \cdot \frac{d}{dx}\left[\int_0^H \rho u \, dy\right] dx$

$\therefore$ Net loss of momentum is

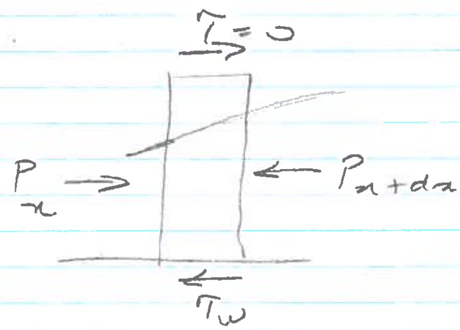
$$\frac{d}{dx}\left[\int_0^H \rho u^2 dy\right] dx - U_\alpha \cdot \frac{d}{dx}\left[\int_0^H \rho u \, dy\right] dx$$

If $\phi = \int_0^H \rho u \, dy$ and $\eta = U_\alpha$, the last expression can be written as

$$\frac{d}{dx}\left[\int_0^H \rho u^2 \, dy\right] dx - \frac{d}{dx}\left[U_\alpha \int_0^H \rho u \, dy\right] dx +$$

$$\frac{dU_\alpha}{dx} \cdot \int_0^H \rho u \, dy \cdot dx$$

forces.



Net force in x-direction

$$-\tau_w \, dx - \frac{d}{dx}\left[\int_0^H P_x \, dy\right] dx$$

For Newtonian fluid, $\tau_w = \mu \left.\frac{du}{dy}\right|_{y=0}$

Assume $P \neq f(y)$

Net Force is $-\mu \left.\frac{du}{dy}\right|_{y=0} dx - H \frac{dP}{dx} \cdot dx$

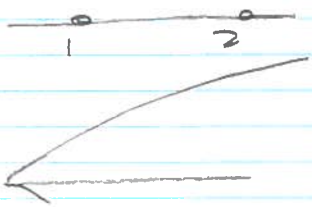
From Newton's law

$$-\mu \left.\frac{du}{dy}\right|_{y=0} dx - H \frac{dP}{dx} dx = -\frac{d}{dx}\left[\int_0^H \rho (U_\alpha - u) u \, dy\right] dx$$

$$+ \frac{dU_\alpha}{dx} dx \cdot \int_0^H \rho u \, dy$$

This is the generalized integral momentum equation of the boundary layer.

Consider a streamline outside the boundary layer, i.e. in the free stream,



Along the streamline, Bernoulli equation is satisfied, i.e

$$\frac{dp}{\rho} + g\,dz + u\,du = 0$$

For constant elevation & $p = P$, $u = U_\alpha$

$$\frac{dP}{\rho} = -U_\alpha\, dU_\alpha$$

$\therefore$ If either of P or U_α is constant, the other variable is also constant. When this is satisfied, the momentum equation can be simplified to

$$\mu \left.\frac{du}{dy}\right|_{y=0} = \frac{d}{dx}\left[\int_0^H \rho(U_\alpha - u)u\,dy\right]$$

The integrand on the r.h.s $= 0$ when $y \geq \delta$

$$\therefore \mu \left.\frac{du}{dy}\right|_{y=0} = \frac{d}{dx}\left[\int_0^\delta \rho(U_\alpha - u)u\,dy\right]$$

Q.E.D.

(b) Bonus question

Given b.c — $\left\{ \begin{array}{lll} y=0 & u=0 & 1, \\ y=\delta & u=U_\alpha & 2, \\ y=\delta & \frac{du}{dy}=0 & 3, \end{array} \right.$

and $u = a + by + cy^2$

apply b.c 1 $\Rightarrow$ $a = 0$

b.c. 3 $\quad \frac{du}{dy} = b + 2cy \Rightarrow 0 = b + 2c\delta$

b.c. 2 $\quad u = by + cy^2 \Rightarrow \underline{U_\alpha} = b\delta + c\delta^2$

solve $\quad -U_\alpha = c\delta^2$ or $c = -U_\alpha/\delta^2$

and $\quad b = \frac{2U_\alpha}{\delta}$

$$\therefore \quad u = \frac{2U_\alpha}{\delta} \cdot y - \frac{U_\alpha}{\delta^2} y^2$$

or $\quad \frac{u}{U_\alpha} = 2\left(\frac{y}{\delta}\right) - \frac{y^2}{\delta^2}$ where $\delta = f(x)$

Let $\phi = \frac{u}{U_\alpha}$ and $\eta = y/\delta$

$$\therefore \quad \phi = 2\eta - \eta^2$$

— Substitute into integral equation

First transform

$$\mu \left.\frac{du}{dy}\right|_{y=0} = \mu \left.\frac{d\left(u/U_\alpha\right) \cdot U_\alpha}{d\left(y/\delta\right) \cdot \delta}\right|_{y=0} = \mu \frac{U_\alpha}{\delta} \left.\frac{d\phi}{d\eta}\right|_{\eta=0}$$

$$\frac{d}{dx}\left[\int_0^{\delta} \rho (U_\alpha - u) u \, dy\right] = \frac{d}{dx}\left[\rho U_\alpha^2 \delta \int_{0/\delta}^{\delta/\delta} \left(1 - \frac{u}{U_\alpha}\right)\frac{u}{U_\alpha} \, d\left(\frac{y}{\delta}\right)\right]$$

$$= \frac{d}{dx}\left[\rho U_\alpha^2 \delta \int_0^1 (1-\phi)\phi \, d\eta\right]$$

$$\therefore \quad \mu \frac{U_\alpha}{\delta} \frac{d\phi}{d\eta}\bigg|_{\eta=0} = \frac{d}{dx}\left[\rho U_\alpha^2 \delta \int_0^1 (1-\phi)\phi \, d\eta\right]$$

subst for ϕ

$$\frac{d\phi}{d\eta}\bigg|_{\eta=0} = 2 - 2\eta\bigg|_{\eta=0} = 2$$

$$\int_0^1 (1 - 2\eta + \eta^2)(2\eta - \eta^2) \, d\eta = \frac{2}{15}$$

$$\therefore \quad \mu \frac{U_\alpha}{\delta}(2) = \frac{d}{dx}\left[\rho U_\alpha^2 \delta \frac{2}{15}\right]$$

$$15 \frac{\mu}{\rho U_\alpha} = \delta \frac{d\delta}{dx} = \frac{1}{2} \frac{d\delta^2}{dx}$$

Integrate, subject to at $x = 0$, $\delta = 0$

$$\delta^2 = 30 \frac{\nu x}{U_\alpha} \quad \text{or} \quad \delta = 5.4772 \sqrt{\frac{\nu x}{U_\alpha}}$$

$\longrightarrow$