ENCH 501 F2017 **Quiz #4** Name:______________________________

October 31, 2017 Time Allowed: 45 minutes aJ

The following expression is given to describe the velocity profile in the boundary layer on a flat plate:

u = a + b sin(cy) with the natural conditions y = 0 u = 0, y = δ u = U_α, and y = δ $\frac{du}{dy} = 0$.

Obtain an expression for the displacement thickness (δ_1) as a function of the distance from the leading edge (x), the free stream velocity (U_α) and the fluid density and dynamic viscosity (ρ, μ). Show all steps.

Potentially useful information: The integral momentum equation of the boundary layer is

$$\mu \frac{du}{dy}\bigg|_{y=0} = \frac{d}{dx}\left[\int_0^\delta \rho(U_\alpha - u)u\,dy\right]$$

and $\int sin(au)du = -\frac{1}{a}cos(au) + C$; $\frac{d}{dx}(\sin u) = \cos u\,\frac{du}{dx}$; $\int sin^2 u\,du = ½(u - \sin u \cos u) + C$

ENCH 501 Quiz 4 Solution G. Jaje

Given $u = a + b \sin(cy)$

use b.c. $y = 0$, $u = 0$ $\Rightarrow a = 0$

$$\frac{du}{dy} = bc \cos(cy)$$

at $y = \delta$, $\frac{du}{dy} = 0$ or $0 = bc\cos(c\delta)$

$$\Rightarrow c\delta = \pi/2$$

$$\therefore c = \frac{\pi}{2}\frac{1}{\delta}$$

and at $y = \delta$ $u = U_\alpha$

$$U_\alpha = b \sin\left(\frac{\pi}{2}\right) \quad \text{or} \quad b = U_\alpha.$$

The velocity profile is

$$\frac{u}{U_\alpha} = \sin\left(\frac{\pi}{2}\, y/\delta\right)$$ where $\delta(x)$ is not known

Substitute profile into the integral momentum equation of the boundary layer

$$\mu \left.\frac{du}{dy}\right|_{y=0} = \frac{d}{dx}\left[\int_0^\delta \rho U_\alpha^2 \left(1 - \frac{u}{U_\alpha}\right)\frac{u}{U_\alpha}\, dy\right]$$

$$\mu U_\alpha \left(\frac{\pi}{2\delta}\right)\cos(0) = \rho U_\alpha^2 \frac{d}{dx}\left[\delta \int_0^\delta \left(1 - \frac{u}{U_\alpha}\right)\frac{u}{U_\alpha}\, \frac{dy}{\delta}\right]$$

$$\frac{\pi}{2}\frac{\nu}{U_\alpha \delta} = \frac{d}{dx}\left[\delta \int_0^1 \left(1 - \frac{u}{U_\alpha}\right)\frac{u}{U_\alpha}\, d\left(\frac{y}{\delta}\right)\right]$$

Solve the integral

$$\int_0^1 \left(1 - \sin\left(\frac{\pi}{2}\frac{y}{\delta}\right)\right) \sin\left(\frac{\pi}{2}\frac{y}{\delta}\right) d\left(y/\delta\right) \quad ; \text{ let } \eta = y/\delta$$

$$\int_0^1 \left(\sin\left(\frac{\pi}{2}\eta\right) - \sin^2\left(\frac{\pi}{2}\eta\right)\right) d\eta$$

$$-\frac{2}{\pi}\cos\left(\frac{\pi}{2}\eta\right)\Big|_0^1 - \frac{1}{2}\frac{2}{\pi}\left[\frac{\pi}{2}\eta - \sin\left(\frac{\pi}{2}\eta\right)\cos\left(\frac{\pi}{2}\eta\right)\right]_0^1$$

$$-\frac{2}{\pi}\left[0 - 1\right] - \frac{1}{\pi}\left[\frac{\pi}{2}\right] = \frac{2}{\pi} - \frac{1}{2} = 0.1366$$

Let $\beta = \frac{\pi}{2}\frac{\nu}{U_\alpha}$

Then

$$\beta/\delta = 0.1366\,\frac{d\delta}{dx}$$

$$\frac{\beta}{0.1366}\,dx = \delta\, d\delta = \frac{1}{2}\,d\delta^2 \qquad \text{with } x=0,\ \delta=0$$

integrate

$$\frac{2\beta}{0.1366}\,x = \delta^2 = \cancel{2}\,\frac{\pi}{\cancel{2}}\,\frac{\nu}{U_\alpha}\,\frac{1}{0.1366}\,x$$

$$\delta = 4.795\sqrt{\frac{\nu x}{U_\alpha}}$$

$\longrightarrow$

The displacement thickness, δ_1 is given by (for incompressible fluid)

$$\delta_1 = \int_0^\delta \left(1 - \frac{u}{U_\alpha}\right) dy$$

$$= \delta \int_0^1 \left(1 - \frac{u}{U_\alpha}\right) d(y/\delta)$$

$$= \delta \int_0^1 \left(1 - \sin\left(\tfrac{\pi}{2}\eta\right)\right) d\eta \quad ; \quad \eta = y/\delta$$

$$= \delta \left[\eta - \left(-\frac{2}{\pi}\cos\left(\tfrac{\pi}{2}\eta\right)\right) \right]_0^1$$

$$= \delta \left[1 - 2/\pi \right] = 0.3634\delta$$

Hence

$$\delta_1 = 1.7425 \sqrt{\frac{\nu x}{U_\alpha}}$$

$\longrightarrow$