MULTIPLE DEGREES OF FREEDOM STRUCTURAL DYNAMICS

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CE571 - Earthquake Engineering

Spring 2003

Introduction

This is a short presentation of the principles of structural dynamics applied to systems having several degrees of freedom for use by students in an earthquake engineering introductory course.

The presented material is based, in the great majority, in corresponding sections of the book "*Dinámica Estructural Aplicada al Diseño Sísmico*" by Luis E. Garcia R., Universidad de los Andes, Bogota, Colombia, 1998, 574 p.

Classical solution of the dynamic equilibrium equations

For free vibration, we have the following system of $\mathbf{n}$ differential simultaneous equilibrium equations:

$$[\mathbf{M}]\{\ddot{\mathbf{U}}\}+[\mathbf{K}]\{\mathbf{U}\}=\{\mathbf{0}\} \qquad (1)$$

$[\mathbf{M}]$ and $[\mathbf{K}]$ are the mass and stiffness matrices, respectively, both being positive defined — meaning that for the equilibrium position the potential energy of the system is cero.

We can propose the following type of solution of the simultaneous differential equations:

$$\{\mathbf{U}_i(\mathbf{t})\}=\{\phi^{(i)}\}\,\mathbf{f}_i(\mathbf{t}) \qquad (2)$$

This is a solution that is separable into an amplitude vector, $\{\phi^{(i)}\}$, and a time dependant function, $\mathbf{f}_i(\mathbf{t})$. Deriving Eq. (2) twice against time we obtain the following acceleration equation:

$$\{\ddot{\mathbf{U}}_i(\mathbf{t})\}=\{\phi^{(i)}\}\,\ddot{\mathbf{f}}_i(\mathbf{t}) \qquad (3)$$

Substituting (2) and (3) in (1) we obtain:

$$[\mathbf{M}]\{\phi^{(i)}\}\ddot{\mathbf{f}}_i(\mathbf{t})+[\mathbf{K}]\{\phi^{(i)}\}\mathbf{f}_i(\mathbf{t})=\{\mathbf{0}\} \qquad (4)$$

Eq. (4) can be seen as $\mathbf{n}$ equations of the type:

$$\left(\sum_{j=1}^{n}\mathbf{m}_{ij}\phi_j^{(i)}\right)\ddot{\mathbf{f}}_i(\mathbf{t})+\left(\sum_{j=1}^{n}\mathbf{k}_{ij}\phi_j^{(i)}\right)\mathbf{f}_i(\mathbf{t})=\mathbf{0} \qquad (5)$$

This opens the door, in the $\mathbf{i}^{th}$ equation, for the use of the classical differential equation solution of separation of variables:

$$-\frac{\ddot{f}_i(t)}{f_i(t)} = \frac{\sum_{j=1}^{n} k_{ij}\phi_j^{(i)}}{\sum_{j=1}^{n} m_{ij}\phi_j^{(i)}} \tag{6}$$

In Eq. (6) we can see that the left side depends on time while the right doesn't. This means that both sides are equal to a constant that we arbitrarily name as ω_i^2. Therefore, the equation can be converted into two equations, one of them being dependent on time and the other not; and both, in turn, equal to the constant ω_i^2:

$$\ddot{f}_i(t) + \omega_i^2 f_i(t) = 0 \tag{7}$$

and

$$\sum_{j=1}^{n}\left(k_{ij} - \omega_i^2 m_{ij}\right)\phi_j^{(i)} = 0 \tag{8}$$

The solution of Eq. (7) is of type:

$$f_i(t) = A_i \sin\omega_i t + B_i \cos\omega_i t \tag{9}$$

Where A_i and B_i are constants that depend on the initial conditions and represent the amplitude of the harmonic movement, with ω_i being the natural frequency in radians per second. The values that ω_i can take are obtained from Eq. (8), that presented in matrix form is:

$$\left[[K] - \omega_i^2 [M]\right]\left\{\phi^{(i)}\right\} = \{0\} \tag{10}$$

Eq. (10) corresponds to a homogeneous simultaneous equation system, whose non-trivial solution only exists if the determinant of the coefficient matrix is cero:

$$\Delta = \left|[K] - \omega_i^2 [M]\right| = 0 \tag{11}$$

Δ is called the *characteristic determinant* of the simultaneous equation system. Expanding the determinant, we can find a polynomial of degree $2n$ that has ω^2 as the variable. This polynomial is called the *characteristic equation* or frequency equation. The n roots of this equation are the natural frequencies of the system, or *eigenvalues*. Since both $[M]$ and $[K]$ are positively defined, it is possible to prove that the roots of the characteristic equation are always real and positive. These root are ordered from minor to largest as:

$$\omega_1^2 \le \omega_2^2 \le \omega_3^2 \le \cdots \le \omega_n^2 \tag{12}$$

The square roots of them are called the natural frequencies, in radians per second. The smaller frequency, ω_1, is called *fundamental frequency*.

So far, we have solved half of the problem. Now we must seek the values of the amplitudes of the harmonic movement $\{\phi^{(i)}\}$, by replacing the values of ω_i^2 in Eq. (10) This leads us to $\mathbf{n}$ systems of simultaneous equation of the type:

$$\left[[\mathbf{K}]-\omega_r^2[\mathbf{M}]\right]\left\{\phi^{(r)}\right\}=\{\mathbf{0}\} \quad \mathbf{r}=\mathbf{1},\mathbf{2},\cdots,\mathbf{n} \tag{13}$$

Where for each value of ω_r there is a vector $\{\phi^{(r)}\}$ corresponding to the non-trivial solution of the simultaneous equation system presented in Eq. (13). $\{\phi^{(r)}\}$ is known as the *characteristic vector*, or *vibration mode* or *"eigenvector"*. This vector is composed by $\mathbf{n}$ elements $\phi_i^{(r)}$, all of them real numbers having no definite value — in the strict sense — since for any real scalar α_r, $\alpha_r\{\phi^{(r)}\}$ is also a solution of the simultaneous equation system described by Eq. (13).

This means that the ratio between the different terms of vector $\{\phi^{(r)}\}$ is fixed and unique. Then, for each frequency ω_r we have a vector $\{\phi^{(r)}\}$ that has a definite shape but arbitrary amplitude. Since there is a possibility that two, or more, frequencies be equal any linear combination of the corresponding modes is also a mode.

By assigning a definite value, such as one (= 1), to any of the terms of vector $\{\phi^{(r)}\}$ the remaining $\mathbf{n-1}$ terms are defined in a unique manner. This process is called normalization and the resulting vectors are called *normal modes*. The following normalization has been popular:

$$\left\{\phi^{(r)}\right\}^{T}\left\{\phi^{(r)}\right\}=\mathbf{1} \tag{14}$$

Sometimes it is convenient to normalize the modes with respect to the mass matrix $[\mathbf{M}]$:

$$\left\{\phi^{(r)}\right\}^{T}[\mathbf{M}]\left\{\phi^{(r)}\right\}=\mathbf{1} \tag{15}$$

This last normalization is called *orthonormal* and is used widely because it simplifies some of the numerical work when solving for the dynamic response of the system.

The different modes are collected in a single matrix, called *modal matrix*, $[\Phi]$, having dimensions of $\mathbf{n}$ by $\mathbf{n}$, and in which each column corresponds to a mode with the first mode located in the first column and the rest being placed in ascending order.

$$[\Phi] = \left[\left\{\phi^{(1)}\right\} \middle| \left\{\phi^{(2)}\right\} \middle| \cdots \middle| \left\{\phi^{(n)}\right\} \right] \tag{16}$$

Vibration modes are properties of the system in the same fashion that the natural frequencies are. They depend on the mass and stiffness properties of the system. Each mode can be excited independently from the other modes. If the initial conditions of the movement are selected in such a way that they excite only mode ($\mathbf{r}$), $\{\phi^{(r)}\}$, the movement of the set of masses that comprise the system will be totally proportional to the shape of the mode and the system will respond in harmonic oscillation with a frequency that is the corresponding frequency of that particular mode ω_r, in radians/second.

Based on this, the general movement of an $\mathbf{n}$ degree of freedom system may be represented by the superposition of the response of the individual modes, each multiplied by constants that depend on the initial conditions, or on the excitation characteristics if we are dealing with forced excitation. These constants indicate the degree of participation of each individual mode in the total response. The total response, for the case of free vibration, is described using a set of new degrees of freedom, η_i, in such a manner that they relate to the original degrees of freedom employed to establish equilibrium through the following relationship:

$$\{\mathbf{U(t)}\} = [\Phi]\{\eta(\mathbf{t})\} \tag{17}$$

For free vibration each of the terms of vector $\{\eta(\mathbf{t})\}$ have the following form:

$$\eta_i(\mathbf{t}) = \mathbf{A}_i \ \mathbf{sin}(\omega_i \mathbf{t}) + \mathbf{B}_i \ \mathbf{cos}(\omega_i \mathbf{t}) \tag{18}$$

Eq. (17) may be transformed into:

$$\begin{aligned} \{\mathbf{U(t)}\} &= [\Phi]\{\eta(\mathbf{t})\} \\ &= [\Phi]\{\mathbf{A}_i \ \mathbf{sin}(\omega_i \mathbf{t}) + \mathbf{B}_i \ \mathbf{cos}(\omega_i \mathbf{t})\} \\ &= \sum_{i=1}^{n}\left(\left\{\phi^{(i)}\right\}\mathbf{A}_i \ \mathbf{sin}(\omega_i \mathbf{t})\right) + \sum_{i=1}^{n}\left(\left\{\phi^{(i)}\right\}\mathbf{B}_i \ \mathbf{cos}(\omega_i \mathbf{t})\right) \end{aligned} \tag{19}$$

Deriving against time Eq. (19) we can obtain the velocity response of each of the original degrees of freedom:

$$\{\dot{\mathbf{U}}(\mathbf{t})\} = \sum_{i=1}^{n}\left(\left\{\phi^{(i)}\right\}\mathbf{A}_i\omega_i \ \mathbf{cos}(\omega_i \mathbf{t})\right) - \sum_{i=1}^{n}\left(\left\{\phi^{(i)}\right\}\mathbf{B}_i\omega_i \ \mathbf{sin}(\omega_i \mathbf{t})\right) \tag{20}$$

If the initial conditions are defined as $\{\mathbf{U_o}\}$ and $\{\dot{\mathbf{U}}_\mathbf{o}\}$ for displacement and velocity respectively, we can compute the following constants $\mathbf{A_i}$ and $\mathbf{B_i}$:

$$\{\mathbf{U_o}\} = \sum_{i=1}^{n}\left(\left\{\phi^{(i)}\right\}\mathbf{B}_i\right) \quad \text{and} \quad \{\dot{\mathbf{U}}_\mathbf{o}\} = \sum_{i=1}^{n}\left(\left\{\phi^{(i)}\right\}\mathbf{A}_i\omega_i\right) \qquad (21)$$

Thus, it is possible to define two systems of simultaneous equations that have as unknowns the values of $\mathbf{B_i}$ and $\mathbf{A_i}\,\omega_i$. Once the unknowns are solved for, the solution for the free vibration dynamic response of the system is obtained.

Example 1

For the building shown in Fig. 1, we are interested in the response in the direction of the numeric reference axes. The lateral stiffness of each one of the stories is the same and can be defined as **k**. *The mass of the two lower stories is twice that of the roof. The roof mass is defined as* **m**.

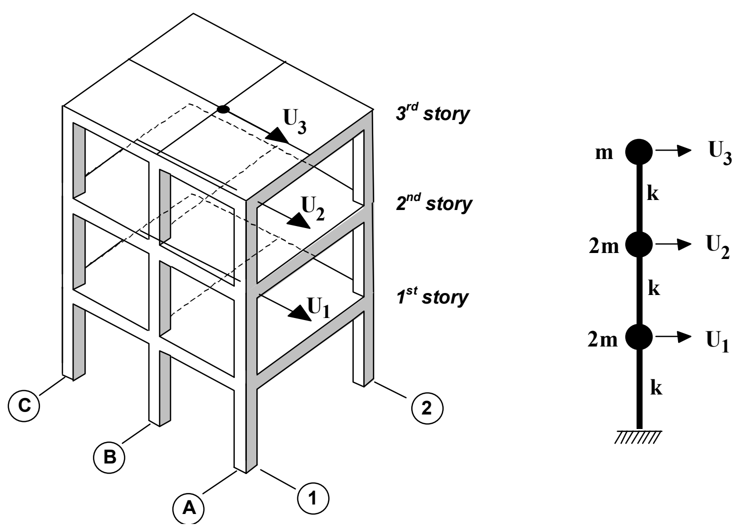


Fig. 1 - Example-1 building

The mass matrix of the structure is:

$$
\begin{array}{cc}
 & \downarrow \textbf{dof} \\
[\mathbf{M}] = \begin{bmatrix} \mathbf{m} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{2m} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{2m} \end{bmatrix} & \begin{array}{c} \mathbf{U_3} \\ \mathbf{U_2} \\ \mathbf{U_1} \end{array}
\end{array}
$$

The stiffness matrix, obtained from equilibrium of each mass is:

$$[\mathbf{K}] = \begin{array}{c} \\ \left[\begin{array}{c|c|c} \mathbf{k} & -\mathbf{k} & \mathbf{0} \\ \hline -\mathbf{k} & \mathbf{2k} & -\mathbf{k} \\ \hline \mathbf{0} & -\mathbf{k} & \mathbf{2k} \end{array}\right] \end{array} \begin{array}{c} \downarrow \textbf{dof} \\ \mathbf{U_3} \\ \mathbf{U_2} \\ \mathbf{U_1} \end{array}$$

The dynamic equilibrium equations are then:

$$\left[\begin{array}{c|c|c} \mathbf{m} & \mathbf{0} & \mathbf{0} \\ \hline \mathbf{0} & \mathbf{2m} & \mathbf{0} \\ \hline \mathbf{0} & \mathbf{0} & \mathbf{2m} \end{array}\right] \left\{\begin{array}{c} \ddot{\mathbf{U}}_3 \\ \hline \ddot{\mathbf{U}}_2 \\ \hline \ddot{\mathbf{U}}_1 \end{array}\right\} + \left[\begin{array}{c|c|c} \mathbf{k} & -\mathbf{k} & \mathbf{0} \\ \hline -\mathbf{k} & \mathbf{2k} & -\mathbf{k} \\ \hline \mathbf{0} & -\mathbf{k} & \mathbf{2k} \end{array}\right] \left\{\begin{array}{c} \mathbf{U_3} \\ \hline \mathbf{U_2} \\ \hline \mathbf{U_1} \end{array}\right\} = \left\{\begin{array}{c} \mathbf{0} \\ \hline \mathbf{0} \\ \hline \mathbf{0} \end{array}\right\}$$

We now proceed to find the solution of the free vibration response of the system for different initial conditions. From Eq. (11) we have:

$$\Delta = \left|[\mathbf{K}] - \omega_i^2 [\mathbf{M}]\right| = \mathbf{0}$$

After replacing **[K]** and **[M]** we obtain the following determinant:

$$\Delta = \left|\begin{array}{c|c|c} \mathbf{k} - \omega^2\mathbf{m} & -\mathbf{k} & \mathbf{0} \\ \hline -\mathbf{k} & \mathbf{2k} - \omega^2\,\mathbf{2m} & -\mathbf{k} \\ \hline \mathbf{0} & -\mathbf{k} & \mathbf{2k} - \omega^2\,\mathbf{2m} \end{array}\right| = \mathbf{0}$$

Expanding this determinant, we obtain the following characteristic equation:

$$\Delta = \mathbf{4m^3\omega^6 - 12km^2\omega^4 + 9k^2m\omega^2 - k^3 = 0}$$

After dividing all terms of the characteristic equation by $\mathbf{4m^3}$ we obtain:

$$\omega^6 - 3\frac{\mathbf{k}}{\mathbf{m}}\omega^4 + \frac{\mathbf{9}}{\mathbf{4}}\frac{\mathbf{k}^2}{\mathbf{m}^2}\omega^2 - \frac{\mathbf{1}}{\mathbf{4}}\frac{\mathbf{k}^3}{\mathbf{m}^3} = \mathbf{0}$$

A simple inspection of the equation tell us that $\omega^2 = \mathbf{k/m}$ is a root, and by using synthetic division, we transform the characteristic equation into:

$$\left(\omega^2 - \frac{\mathbf{k}}{\mathbf{m}}\right)\left(\mathbf{4m^3\omega^4 - 8km^2\omega^2 + k^2m}\right) = \mathbf{0}$$

Solving the second-degree equation contained in the second term of the previous equation, we obtain:

$$\omega^2 = \frac{8km^2 \pm \sqrt{64k^2m^4 - 16k^2m^4}}{8m^3} = \frac{k}{m}\left[1 \pm \frac{\sqrt{3}}{2}\right] = \left\langle \begin{array}{l} 1.866\dfrac{k}{m} \\ 0.134\dfrac{k}{m} \end{array} \right.$$

Then, the natural frequencies of the building — properly ordered — are:

$$\omega_1^2 = 0.134\frac{k}{m} \qquad \omega_2^2 = \frac{k}{m} \qquad \omega_3^2 = 1.866\frac{k}{m}$$

Now, by using Eq. (13) we can obtain the vibration modes by going back to the characteristic determinant:

$$\left[[K] - \omega_r{}^2 [M]\right]\left\{\phi^{(r)}\right\} = \{0\} \quad r = 1, 2 \text{ and } 3$$

Replacing here the mass and stiffness matrices, we obtain the following set of homogeneous simultaneous equations:

$$\left[\begin{array}{c|c|c} k - \omega_r^2 m & -k & 0 \\ \hline -k & 2k - \omega_r^2 2m & -k \\ \hline 0 & -k & 2k - \omega_r^2 2m \end{array}\right]\left\{\begin{array}{c} \phi_3^{(r)} \\ \hline \phi_2^{(r)} \\ \hline \phi_1^{(r)} \end{array}\right\} = \left\{\begin{array}{c} 0 \\ \hline 0 \\ \hline 0 \end{array}\right\}$$

Expanding the product, we see the system in the classical simultaneous equation format:

$$\begin{array}{rcccccl} \left(k - \omega_r^2 m\right)\phi_3^{(r)} & - & k\phi_2^{(r)} & & & = & 0 \\ -k\phi_3^{(r)} & + & \left(2k - \omega_r^2 2m\right)\phi_2^{(r)} & - & k\phi_1^{(r)} & = & 0 \\ & - & k\phi_2^{(r)} & + & \left(2k - \omega_r^2 2m\right)\phi_1^{(r)} & = & 0 \end{array}$$

From the third equation, we can see that, in this case, the ratio between the second unknown and the first unknown is:

$$\frac{\phi_2^{(r)}}{\phi_1^{(r)}} = \frac{2k - \omega_r^2 2m}{k}$$

Now replacing the third equation into the second, we obtain the following ratio between the third unknown and the first unknown:

$$\frac{\phi_3^{(r)}}{\phi_1^{(r)}} = \frac{\left(2k - \omega_r^2\,2m\right)^2}{k^2} - 1$$

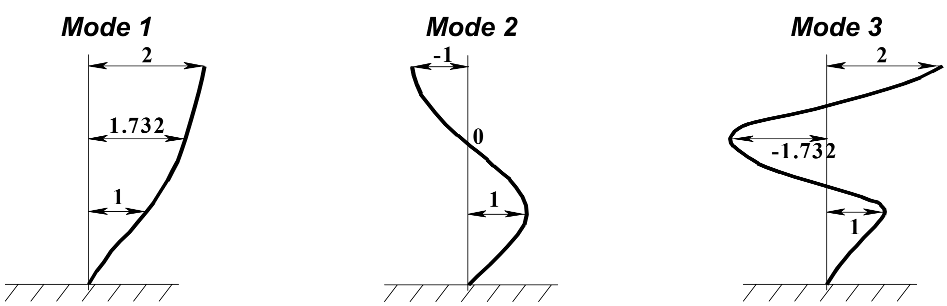
These two ratios are fixed for any value of ω_i^2. We now replace the values of ω_i^2 obtained previously and the values of the unknowns are found for each case:

	ω_1^2	ω_2^2	ω_3^2
$\frac{\phi_2}{\phi_1}$	**1.732**	**0**	**-1.732**
$\frac{\phi_3}{\phi_1}$	**2**	**-1**	**2**

We may assign any arbitrary value to the ϕ_1 term and thus from the obtained ratios compute the other two values of the terms of the mode. We choose, arbitrarily again, a value of one for ϕ_1. By doing so, the modes are defined as:

$$\left\{\phi^{(1)}\right\} = \left\{\begin{array}{c} 2 \\ \hline 1.732 \\ \hline 1 \end{array}\right\} \qquad \left\{\phi^{(2)}\right\} = \left\{\begin{array}{c} -1 \\ \hline 0 \\ \hline 1 \end{array}\right\} \qquad \left\{\phi^{(3)}\right\} = \left\{\begin{array}{c} 2 \\ \hline -1.732 \\ \hline 1 \end{array}\right\}$$

Corresponding, graphically, to:



$$\omega_1^2 = 0.134\frac{k}{m} \qquad \omega_2^2 = \frac{k}{m} \qquad \omega_3^2 = 1.866\frac{k}{m}$$

We now change the normalization of the modes in such a way that they comply with Eq. (15) to obtain orthonormal modes:

$$\left\{\phi^{(r)}\right\}^T\left[\mathbf{M}\right]\left\{\phi^{(r)}\right\}=1$$

Mode 1

$$\left\{2 \mid \sqrt{3} \mid 1\right\}\begin{bmatrix} m & 0 & 0 \\ 0 & 2m & 0 \\ 0 & 0 & 2m \end{bmatrix}\begin{Bmatrix} 2 \\ \sqrt{3} \\ 1 \end{Bmatrix} = 12m \quad \Rightarrow \left\{\phi^{(1)}\right\} = \frac{1}{\sqrt{12m}}\begin{Bmatrix} 2 \\ \sqrt{3} \\ 1 \end{Bmatrix} = \begin{Bmatrix} 0.5774/\sqrt{m} \\ 0.5000/\sqrt{m} \\ 0.2887/\sqrt{m} \end{Bmatrix}$$

Mode 2

$$\left\{-1 \mid 0 \mid 1\right\}\begin{bmatrix} m & 0 & 0 \\ 0 & 2m & 0 \\ 0 & 0 & 2m \end{bmatrix}\begin{Bmatrix} -1 \\ 0 \\ 1 \end{Bmatrix} = 3m \quad \Rightarrow \left\{\phi^{(2)}\right\} = \frac{1}{\sqrt{3m}}\begin{Bmatrix} -1 \\ 0 \\ 1 \end{Bmatrix} = \begin{Bmatrix} -0.5774/\sqrt{m} \\ 0 \\ 0.5774/\sqrt{m} \end{Bmatrix}$$

Mode 3

$$\left\{2 \mid -\sqrt{3} \mid 1\right\}\begin{bmatrix} m & 0 & 0 \\ 0 & 2m & 0 \\ 0 & 0 & 2m \end{bmatrix}\begin{Bmatrix} 2 \\ -\sqrt{3} \\ 1 \end{Bmatrix} = 12m \quad \Rightarrow \left\{\phi^{(3)}\right\} = \frac{1}{\sqrt{12m}}\begin{Bmatrix} 2 \\ -\sqrt{3} \\ 1 \end{Bmatrix} = \begin{Bmatrix} 0.5774/\sqrt{m} \\ -0.5000/\sqrt{m} \\ 0.2887/\sqrt{m} \end{Bmatrix}$$

The modal matrix is then:

$$\left[\Phi\right] = \frac{1}{\sqrt{m}}\begin{bmatrix} 0.5774 & -0.5774 & 0.5774 \\ 0.5000 & 0 & -0.5000 \\ 0.2887 & 0.5774 & 0.2887 \end{bmatrix}$$

■

Orthogonality of the natural modes

Each mode, independently, is obtained from solving the homogenous system of simultaneous equation depicted in Eq. (13), which is repeated here for convenience.

$$\left[\left[\mathbf{K}\right]-\omega_r^2\left[\mathbf{M}\right]\right]\left\{\phi^{(r)}\right\}=\left\{0\right\} \quad \mathbf{r}=1,2,\cdots,\mathbf{n} \tag{13}$$

By using mode $\mathbf{r}$, we have the following equation:

$$[\mathbf{K}]\{\phi^{(r)}\} = \omega_r^2 [\mathbf{M}]\{\phi^{(r)}\} \quad (22)$$

By pre-multiplying by a different mode transposed, say **s**, we obtain:

$$\{\phi^{(s)}\}^T [\mathbf{K}]\{\phi^{(r)}\} = \omega_r^2 \{\phi^{(s)}\}^T [\mathbf{M}]\{\phi^{(r)}\} \quad (23)$$

If we had initiated the process in Eq. (22) with mode number **s** instead, and afterwards we had pre-multiplied by mode **r** transposed, we would had obtained:

$$\{\phi^{(r)}\}^T [\mathbf{K}]\{\phi^{(s)}\} = \omega_s^2 \{\phi^{(r)}\}^T [\mathbf{M}]\{\phi^{(s)}\} \quad (24)$$

Applying to Eq. (24) the principle that states that $([\mathbf{A}][\mathbf{B}][\mathbf{C}])^T = [\mathbf{C}]^T[\mathbf{B}]^T[\mathbf{A}]^T$, and knowing that $[\mathbf{K}] = [\mathbf{K}]^T$ and $[\mathbf{M}] = [\mathbf{M}]^T$ because they are symmetric, we obtain:

$$\{\phi^{(s)}\}^T [\mathbf{K}]\{\phi^{(r)}\} = \omega_s^2 \{\phi^{(s)}\}^T [\mathbf{M}]\{\phi^{(r)}\} \quad (25)$$

Subtracting Eq. (25) from Eq. (23) we find:

$$\mathbf{0} = \left(\omega_r^2 - \omega_s^2\right)\{\phi^{(s)}\}^T [\mathbf{M}]\{\phi^{(r)}\} \quad (26)$$

In most cases, the values of the natural frequencies are different, then:

$$\left(\omega_r^2 - \omega_s^2\right) \neq \mathbf{0} \quad (27)$$

Therefore, the following product has to be cero:

$$\{\phi^{(s)}\}^T [\mathbf{M}]\{\phi^{(r)}\} = \mathbf{0} \quad (28)$$

We can obtain a similar result by starting from Eq. (22) expressed as:

$$\frac{\mathbf{1}}{\omega_r^2}[\mathbf{K}]\{\phi^{(r)}\} = [\mathbf{M}]\{\phi^{(r)}\} \quad (29)$$

Thus:

$$\{\phi^{(s)}\}^T [\mathbf{K}]\{\phi^{(r)}\} = \mathbf{0} \quad (30)$$

Summarizing, the *orthogonality principle* tell us that if the modes are orthonormally normalized as defined by Eq. (15), then:

$$\left\{\phi^{(s)}\right\}^T\left[\mathbf{M}\right]\left\{\phi^{(r)}\right\}=\begin{cases}\mathbf{1} & \textbf{if} \quad \mathbf{s=r}\\ \mathbf{0} & \textbf{if} \quad \mathbf{s\neq r}\end{cases} \tag{31}$$

and

$$\left\{\phi^{(s)}\right\}^T\left[\mathbf{K}\right]\left\{\phi^{(r)}\right\}=\begin{cases}\omega_s^2 & \textbf{if} \quad \mathbf{s=r}\\ \mathbf{0} & \textbf{if} \quad \mathbf{s\neq r}\end{cases} \tag{32}$$

By expressing the orthogonality property through the mass and stiffness properties, as described by the mass $[\mathbf{M}]$ and stiffness $[\mathbf{K}]$ matrices, the modal vectors constitute a linearly independent set. This means that a vector with any configuration can always be expressed as a linear combination of the modes and; therefore, the modes can describe any possible movement of the system.

If the modes are not normalized in such a way that they are orthonormal, then for the case of $\mathbf{r} = \mathbf{s}$ in Eqs. (31) and (32) the values would not be those shown there, but would be non-cero in all cases. The relationship shown in Eq. (33) and known as *Rayleigh quotient*, is true even for approximations of the shape of the mode, and is the basis of Rayleigh's method for obtaining vibration frequencies and modes.

$$\frac{\left\{\phi^{(s)}\right\}^T\left[\mathbf{K}\right]\left\{\phi^{(s)}\right\}}{\left\{\phi^{(s)}\right\}^T\left[\mathbf{M}\right]\left\{\phi^{(s)}\right\}}=\omega_s^2 \tag{33}$$

Uncoupling of the dynamic equilibrium equations

If after normalizing the modes through Eq. (15), thus making them orthonormal, and building a $[\Phi]$ matrix as described by Eq. (16); we can employ this matrix to perform a coordinate transformation as follows:

$$\left\{\mathbf{U}\right\}=\left[\Phi\right]\left\{\eta\right\} \tag{34}$$

By deriving twice against time, we obtain:

$$\left\{\ddot{\mathbf{U}}\right\}=\left[\Phi\right]\left\{\ddot{\eta}\right\} \tag{35}$$

On the other hand, Eq. (13) tell us that eigenvalues problem can be transformed into:

$$\left[\mathbf{K}\right]\left[\Phi\right]=\left[\mathbf{M}\right]\left[\Phi\right]\left[\omega^2\right] \tag{36}$$

Where $[\omega^2]$ is a diagonal matrix. Pre-multiplying both sides of Eq. (36) by $[\Phi]^T$, we obtain:

$$[\Phi]^T[\mathbf{K}][\Phi]=[\Phi]^T[\mathbf{M}][\Phi]\left[\omega^2\right] \tag{37}$$

Then by using the definition of orthonormal normalization:

$$[\Phi]^T[\mathbf{M}][\Phi]=[\mathbf{I}] \tag{38}$$

We know that by using the orthogonality principle, the following is true:

$$[\Phi]^T[\mathbf{K}][\Phi]=\left[\omega^2\right] \tag{39}$$

Now, replacing Eqs. (34) and (35) in Eq. (1), we obtain:

$$[\mathbf{M}][\Phi]\{\ddot{\eta}\}+[\mathbf{K}][\Phi]\{\eta\}=\{\mathbf{0}\} \tag{40}$$

Pre-multiply by $[\Phi]^T$ the following result is obtained:

$$\underbrace{[\Phi]^T[\mathbf{M}][\Phi]}_{[\,\mathbf{I}\,]}\{\ddot{\eta}\}+\underbrace{[\Phi]^T[\mathbf{K}][\Phi]}_{\left[\omega^2\right]}\{\eta\}=\{\mathbf{0}\} \tag{41}$$

Then the system has been transformed to:

$$[\,\mathbf{I}\,]\{\ddot{\eta}\}+\left[\omega^2\right]\{\eta\}=\{\mathbf{0}\} \tag{42}$$

Because both $[\,\mathbf{I}\,]$ and $[\omega^2]$ are diagonal matrices, we have been able to uncouple the system. This means that we have transformed a system of $\mathbf{n}$ simultaneous differential equations into $\mathbf{n}$ independent single-degree of freedom differential equations of the type:

$$\ddot{\eta}_i+\omega_i^2\eta_i=\mathbf{0} \tag{43}$$

For free vibration Eq. (43) can be easily solved as the sum of a sine and a cosine affected by amplitude constants that depend solely on the initial conditions of the movement. Once the response in time of each one of the *generalized degrees of freedom*, η_i, is obtained, the response of the structure is the superposition of the individual contribution from each mode:

$$\{\mathbf{U}\}=[\Phi]\{\eta\}=\sum_{i=1}^{n}\left(\left\{\phi^{(i)}\right\}\eta_i(\mathbf{t})\right)=\left\{\phi^{(1)}\right\}\eta_1(\mathbf{t})+\left\{\phi^{(2)}\right\}\eta_2(\mathbf{t})+\ \cdots\ +\left\{\phi^{(n)}\right\}\eta_n(\mathbf{t}) \tag{44}$$

We have just transformed the coordinate system of the dynamical equilibrium equations, from the system employed to state it, $\{\mathbf{U}\}$, to a generalized coordinates system, $\{\eta\}$, where each degree of freedom acts independently and in turn affects all the original degrees of freedom in such a way that they respond in an harmonic fashion as prescribed by the corresponding mode.

Example 2

Uncouple the dynamic system of Example 1 using matrix **[Φ]**. *From Example 1 have the following information.*

Mass matrix:

$$\begin{array}{cc} & \downarrow \textbf{dof} \\ [\mathbf{M}] = \begin{bmatrix} \mathbf{m} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{2m} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{2m} \end{bmatrix} & \begin{array}{c} \mathbf{U_3} \\ \mathbf{U_2} \\ \mathbf{U_1} \end{array} \end{array}$$

Stiffness matrix:

$$\begin{array}{cc} & \downarrow \textbf{dof} \\ [\mathbf{K}] = \begin{bmatrix} \mathbf{k} & \mathbf{-k} & \mathbf{0} \\ \mathbf{-k} & \mathbf{2k} & \mathbf{-k} \\ \mathbf{0} & \mathbf{-k} & \mathbf{2k} \end{bmatrix} & \begin{array}{c} \mathbf{U_3} \\ \mathbf{U_2} \\ \mathbf{U_1} \end{array} \end{array}$$

Dynamic equilibrium equations:

$$\begin{bmatrix} \mathbf{m} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{2m} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{2m} \end{bmatrix} \begin{Bmatrix} \ddot{\mathbf{U}}_3 \\ \ddot{\mathbf{U}}_2 \\ \ddot{\mathbf{U}}_1 \end{Bmatrix} + \begin{bmatrix} \mathbf{k} & \mathbf{-k} & \mathbf{0} \\ \mathbf{-k} & \mathbf{2k} & \mathbf{-k} \\ \mathbf{0} & \mathbf{-k} & \mathbf{2k} \end{bmatrix} \begin{Bmatrix} \mathbf{U_3} \\ \mathbf{U_2} \\ \mathbf{U_1} \end{Bmatrix} = \begin{Bmatrix} \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \end{Bmatrix}$$

In Example 1 the modal matrix was computed as:

$$[\Phi] = \frac{\mathbf{1}}{\sqrt{\mathbf{m}}} \begin{bmatrix} \mathbf{0.5774} & \mathbf{-0.5774} & \mathbf{0.5774} \\ \mathbf{0.5000} & \mathbf{0} & \mathbf{-0.5000} \\ \mathbf{0.2887} & \mathbf{0.5774} & \mathbf{0.2887} \end{bmatrix}$$

In order to uncouple the system, the following operations are performed:

$$[\Phi]^T[M][\Phi] = \frac{1}{\sqrt{m}}\begin{bmatrix} 0.5774 & 0.5000 & 0.2887 \\ -0.5774 & 0 & 0.5774 \\ 0.5774 & -0.5000 & 0.2887 \end{bmatrix}\begin{bmatrix} m & 0 & 0 \\ 0 & 2m & 0 \\ 0 & 0 & 2m \end{bmatrix}\frac{1}{\sqrt{m}}\begin{bmatrix} 0.5774 & -0.5774 & 0.5774 \\ 0.5000 & 0 & -0.5000 \\ 0.2887 & 0.5774 & 0.2887 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

and

$$[\Phi]^T[K][\Phi] = \frac{1}{\sqrt{m}}\begin{bmatrix} 0.5774 & 0.5000 & 0.2887 \\ -0.5774 & 0 & 0.5774 \\ 0.5774 & -0.5000 & 0.2887 \end{bmatrix}\begin{bmatrix} k & -k & 0 \\ -k & 2k & -k \\ 0 & -k & 2k \end{bmatrix}\frac{1}{\sqrt{m}}\begin{bmatrix} 0.5774 & -0.5774 & 0.5774 \\ 0.5000 & 0 & -0.5000 \\ 0.2887 & 0.5774 & 0.2887 \end{bmatrix}$$

$$= \frac{k}{m}\begin{bmatrix} 0.134 & 0 & 0 \\ 0 & 1.000 & 0 \\ 0 & 0 & 1.866 \end{bmatrix}$$

The uncoupled equations are:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{Bmatrix} \ddot{\eta}_1 \\ \ddot{\eta}_2 \\ \ddot{\eta}_3 \end{Bmatrix} + \frac{k}{m}\begin{bmatrix} 0.134 & 0 & 0 \\ 0 & 1.000 & 0 \\ 0 & 0 & 1.866 \end{bmatrix}\begin{Bmatrix} \eta_1 \\ \eta_2 \\ \eta_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$$

Or seen as three independent differential equations:

$$\ddot{\eta}_1 + 0.134\frac{k}{m}\eta_1 = 0$$

$$\ddot{\eta}_2 + 1.000\frac{k}{m}\eta_2 = 0$$

$$\ddot{\eta}_3 + 1.866\frac{k}{m}\eta_3 = 0$$

■

Free vibration with initial conditions

We have already stated that a general solution of the multi-degree of freedom system, under free vibration, is possible as a superposition of the response of the uncoupled degrees of freedom η_i. These uncoupled degrees of freedom are linked to the degrees of freedom employed to state equilibrium by:

$$\{U(t)\} = [\Phi]\{\eta(t)\} \qquad (45)$$

The elements of vector $\{\eta(t)\}$ have the following form when there is no damping:

$$\eta_i(t) = A_i \sin\omega_i t + B_i \cos\omega_i t \tag{46}$$

Then, Eq. (45) converts into:

$$\{U(t)\} = [\Phi]\{\eta(t)\} = [\Phi]\{A\,\mathrm{sen}\omega t\} + [\Phi]\{B\cos\omega t\} \tag{47}$$

By deriving Eq. (46) against time we obtain:

$$\dot{\eta}_i(t) = \omega_i A_i \cos\omega_i t + (-\omega_i B_i)\sin\omega_i t \tag{48}$$

and

$$\{\dot{U}(t)\} = [\Phi]\{\dot{\eta}(t)\} = [\Phi]\{\omega A\cos\omega t\} + [\Phi]\{-\omega B\sin\omega t\} \tag{49}$$

For initial conditions in displacement, $\{U_o\}$, and velocity, $\{\dot{U}_o\}$, then:

$$\{U_o\} = [\Phi]\{\eta(0)\} = [\Phi]\{B\} \tag{50}$$

and

$$\{\dot{U}_o\} = [\Phi]\{\dot{\eta}(0)\} = [\Phi]\{\omega A\} \tag{51}$$

Pre-multiplying Eqs. (50) and (51) by $[\Phi]^T[M]$, we obtain:

$$[\Phi]^T[M]\{U_o\} = [\Phi]^T[M][\Phi]\{B\} = \{B\} \tag{52}$$

and

$$[\Phi]^T[M]\{\dot{U}_o\} = [\Phi]^T[M][\Phi]\{\omega A\} = \{\omega A\} \tag{53}$$

Then, the response in time of the displacements of an undamped system under free vibration conditions, can be described by:

$$\{U(t)\} = [\Phi]\{\eta(t)\} = [\Phi][\Phi]^T[M]\{\dot{U}_o\}\left\{\frac{1}{\omega}\sin\omega t\right\} + [\Phi][\Phi]^T[M]\{U_o\}\{\cos\omega t\} \tag{54}$$

Total response is then the superposition of the response of the individual modes, as:

$$\{U(t)\} = \sum_{i=1}^{n}\left(\{\phi^{(i)}\}\frac{a_i}{\omega_i}\sin\omega_i t\right) + \sum_{i=1}^{n}\left(\{\phi^{(i)}\}b_i\cos\omega_i t\right) \tag{55}$$

Where:

$$a_i = \{\phi^{(i)}\}^T[M]\{\dot{U}_o\} \tag{56}$$

and

$$\mathbf{b_i} = \left\{\phi^{(i)}\right\}^T \left[\mathbf{M}\right]\left\{\mathbf{U_o}\right\} \tag{57}$$

For damped systems, the same deduction can be used, by making the appropriate changes in Eq. (46).

Example 3

For the building in Example 1, find the free vibration response for different cases of initial displacement conditions.

Case (a) - *Suppose a unit displacement at each story of the building at time = 0, without any initial velocity.*

The initial displacement vector is:

$$\left\{\mathbf{U_o}\right\} = \begin{Bmatrix} \mathbf{U_3(0)} \\ \mathbf{U_2(0)} \\ \mathbf{U_1(0)} \end{Bmatrix} = \begin{Bmatrix} \mathbf{1} \\ \mathbf{1} \\ \mathbf{1} \end{Bmatrix}$$

Constants $\mathbf{b_i}$ *are obtained from:*

$$\left\{\mathbf{B}\right\} = \left[\Phi\right]^T \left[\mathbf{M}\right]\left\{\mathbf{U_o}\right\}$$

$$\left\{\mathbf{B}\right\} = \begin{Bmatrix} \mathbf{b_1} \\ \mathbf{b_2} \\ \mathbf{b_3} \end{Bmatrix} = \frac{\mathbf{1}}{\sqrt{\mathbf{m}}} \begin{bmatrix} \mathbf{0.5774} & \mathbf{0.5000} & \mathbf{0.2887} \\ \mathbf{-0.5774} & \mathbf{0} & \mathbf{0.5774} \\ \mathbf{0.5774} & \mathbf{-0.5000} & \mathbf{0.2887} \end{bmatrix} \begin{bmatrix} \mathbf{m} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{2m} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{2m} \end{bmatrix} \begin{Bmatrix} \mathbf{1} \\ \mathbf{1} \\ \mathbf{1} \end{Bmatrix} = \sqrt{\mathbf{m}} \begin{Bmatrix} \mathbf{2.1547} \\ \mathbf{0.5774} \\ \mathbf{0.1548} \end{Bmatrix}$$

Then, the response of the system is described by the following equation:

$$\begin{Bmatrix} \mathbf{U_3} \\ \mathbf{U_2} \\ \mathbf{U_1} \end{Bmatrix} = \begin{Bmatrix} \mathbf{0.5774} \\ \mathbf{0.5000} \\ \mathbf{0.2887} \end{Bmatrix} \mathbf{2.1547\cos\omega_1 t} + \begin{Bmatrix} \mathbf{-0.5774} \\ \mathbf{0} \\ \mathbf{0.5774} \end{Bmatrix} \mathbf{0.5774\cos\omega_2 t} + \begin{Bmatrix} \mathbf{0.5774} \\ \mathbf{-0.5000} \\ \mathbf{0.2887} \end{Bmatrix} \mathbf{0.1547\cos\omega_3 t}$$

$$= \begin{Bmatrix} \mathbf{1.2441} \\ \mathbf{1.0774} \\ \mathbf{0.6221} \end{Bmatrix} \mathbf{\cos\omega_1 t} + \begin{Bmatrix} \mathbf{-0.3333} \\ \mathbf{0} \\ \mathbf{0.3333} \end{Bmatrix} \mathbf{\cos\omega_2 t} + \begin{Bmatrix} \mathbf{0.0893} \\ \mathbf{-0.0774} \\ \mathbf{0.0447} \end{Bmatrix} \mathbf{\cos\omega_3 t}$$

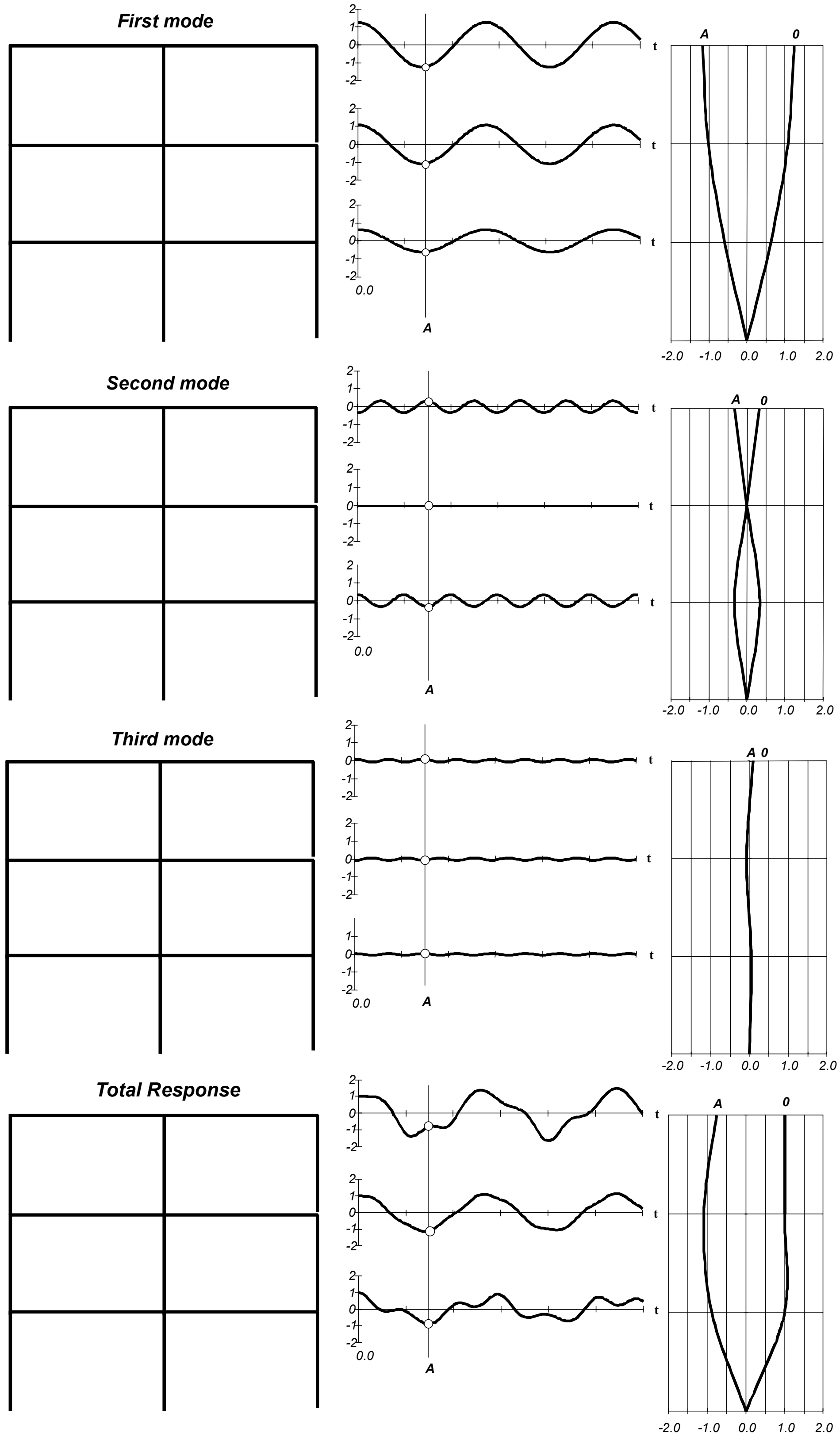


Fig. 2 - Response to initial displacement conditions. Case (a)

It is evident that the response of the system corresponds to the superposition of the individual responses from each mode. Fig. 2 shows the response for each mode and the total response of the building. Supposing that at some instant in time the three responses are in phase, 62.2% would be contributed by the first mode, 33.3% by the second, and 4.5% by the third.

Case (b) - Now lets suppose an initial displacement condition in the shape of the first mode, without initial velocity.

The initial displacement vector is:

$$\{\mathbf{U_o}\} = \begin{Bmatrix} \mathbf{U_3(0)} \\ \hline \mathbf{U_2(0)} \\ \hline \mathbf{U_1(0)} \end{Bmatrix} = \begin{Bmatrix} \mathbf{2} \\ \hline \sqrt{\mathbf{3}} \\ \hline \mathbf{1} \end{Bmatrix}$$

Constants $\mathbf{b_i}$ are obtained from

$$\{\mathbf{B}\} = \begin{Bmatrix} \mathbf{b_1} \\ \hline \mathbf{b_2} \\ \hline \mathbf{b_3} \end{Bmatrix} = \frac{\mathbf{1}}{\sqrt{\mathbf{m}}} \left[\begin{array}{r|r|r} \mathbf{0.5774} & \mathbf{0.5000} & \mathbf{0.2887} \\ \hline \mathbf{-0.5774} & \mathbf{0} & \mathbf{0.5774} \\ \hline \mathbf{0.5774} & \mathbf{-0.5000} & \mathbf{0.2887} \end{array} \right] \left[\begin{array}{c|c|c} \mathbf{m} & \mathbf{0} & \mathbf{0} \\ \hline \mathbf{0} & \mathbf{2m} & \mathbf{0} \\ \hline \mathbf{0} & \mathbf{0} & \mathbf{2m} \end{array} \right] \begin{Bmatrix} \mathbf{2} \\ \hline \sqrt{\mathbf{3}} \\ \hline \mathbf{1} \end{Bmatrix} = \sqrt{\mathbf{m}} \begin{Bmatrix} \mathbf{2\sqrt{3}} \\ \hline \mathbf{0} \\ \hline \mathbf{0} \end{Bmatrix}$$

The response would be described by:

$$\begin{Bmatrix} \mathbf{U_3} \\ \hline \mathbf{U_2} \\ \hline \mathbf{U_1} \end{Bmatrix} = \begin{Bmatrix} \mathbf{0.5774} \\ \hline \mathbf{0.5000} \\ \hline \mathbf{0.2887} \end{Bmatrix} \mathbf{2\sqrt{3}\cos\omega_1 t} = \begin{Bmatrix} \mathbf{2} \\ \hline \sqrt{\mathbf{3}} \\ \hline \mathbf{1} \end{Bmatrix} \mathbf{\cos\omega_1 t}$$

100% of the response is contributed by the first mode alone. The other modes don't contribute.

Case (c) - Now lets suppose an initial displacement in the shape of the second mode without any initial velocity.

The vector of initial displacements is:

$$\{\mathbf{U_o}\} = \begin{Bmatrix} \mathbf{U_3(0)} \\ \hline \mathbf{U_2(0)} \\ \hline \mathbf{U_1(0)} \end{Bmatrix} = \begin{Bmatrix} \mathbf{-1} \\ \hline \mathbf{0} \\ \hline \mathbf{1} \end{Bmatrix}$$

Constants $\mathbf{b_i}$ are obtained from:

$$\{\mathbf{B}\} = [\Phi]^T [\mathbf{M}] \{\mathbf{U_o}\}$$

$$\{\mathbf{B}\} = \begin{Bmatrix} b_1 \\ b_2 \\ b_3 \end{Bmatrix} = \frac{1}{\sqrt{m}} \begin{bmatrix} 0.5774 & 0.5000 & 0.2887 \\ -0.5774 & 0 & 0.5774 \\ 0.5774 & -0.5000 & 0.2887 \end{bmatrix} \begin{bmatrix} m & 0 & 0 \\ 0 & 2m & 0 \\ 0 & 0 & 2m \end{bmatrix} \begin{Bmatrix} -1 \\ 0 \\ 1 \end{Bmatrix} = \sqrt{m} \begin{Bmatrix} 0 \\ \sqrt{3} \\ 0 \end{Bmatrix}$$

The response of the system is described by the following equation:

$$\begin{Bmatrix} U_3 \\ U_2 \\ U_1 \end{Bmatrix} = \begin{Bmatrix} -0.5774 \\ 0 \\ 0.5774 \end{Bmatrix} \sqrt{3} \cos \omega_2 t = \begin{Bmatrix} -1 \\ 0 \\ 1 \end{Bmatrix} \cos \omega_2 t$$

Only the second mode contributes with a 100% of the response.

Case (d) - Now lets suppose an initial displacement in the shape of the second mode without any initial velocity.

The initial displacement vector is:

$$\{\mathbf{U_o}\} = \begin{Bmatrix} U_3(0) \\ U_2(0) \\ U_1(0) \end{Bmatrix} = \begin{Bmatrix} 2 \\ -\sqrt{3} \\ 1 \end{Bmatrix}$$

Constants $\mathbf{b_i}$ are obtained from:

$$\{\mathbf{B}\} = [\Phi]^T [\mathbf{M}] \{\mathbf{U_o}\}$$

$$\{\mathbf{B}\} = \begin{Bmatrix} b_1 \\ b_2 \\ b_3 \end{Bmatrix} = \frac{1}{\sqrt{m}} \begin{bmatrix} 0.5774 & 0.5000 & 0.2887 \\ -0.5774 & 0 & 0.5774 \\ 0.5774 & -0.5000 & 0.2887 \end{bmatrix} \begin{bmatrix} m & 0 & 0 \\ 0 & 2m & 0 \\ 0 & 0 & 2m \end{bmatrix} \begin{Bmatrix} 2 \\ -\sqrt{3} \\ 1 \end{Bmatrix} = \sqrt{m} \begin{Bmatrix} 0 \\ 0 \\ 2\sqrt{3} \end{Bmatrix}$$

Response is described by the following equation:

$$\begin{Bmatrix} \mathbf{U_3} \\ \hline \mathbf{U_2} \\ \hline \mathbf{U_1} \end{Bmatrix} = \begin{Bmatrix} \mathbf{0.5774} \\ \hline \mathbf{-0.5000} \\ \hline \mathbf{0.2887} \end{Bmatrix} \mathbf{2\sqrt{3}\cos\omega_3 t} = \begin{Bmatrix} \mathbf{2} \\ \hline \mathbf{-\sqrt{3}} \\ \hline \mathbf{1} \end{Bmatrix} \mathbf{\cos\omega_3 t}$$

Only the third mode contributes with a 100% of the response.

■

Damped modal analysis

Viscous damping is the damping of choice in many cases for describing the response of single degree of freedom dynamic systems. One of the main reasons for selecting viscous damping is associated with the fact that this type of damping is the most amenable for solving the dynamic equilibrium differential equation. When these concepts are extended to multiple degree of freedom systems, serious shortcomings come into play, because there is not such a clear relationship between the physical phenomena and its mathematical modeling.

A multi-degree of freedom system with viscous damping under free vibration would be described by the following equilibrium equations:

$$[\mathbf{M}]\{\ddot{\mathbf{x}}\} + [\mathbf{C}]\{\dot{\mathbf{x}}\} + [\mathbf{K}]\{\mathbf{x}\} = \{\mathbf{0}\} \tag{58}$$

The force exerted by a viscous damper is proportional to the relative velocity between the two ends of the damper. The procedure to obtain the elements of the damping matrix $[\mathbf{C}]$ consists in imposing a unit velocity to one degree of freedom at a time, while maintaining the velocity of all other degrees of freedom in cero. The internal forces exerted in all degrees of freedom of the structure by the dampers affected by the unit velocity of the selected degree of freedom compose the column of the damping matrix corresponding to the selected degree of freedom.

$$[\mathbf{C}] = \begin{bmatrix} \mathbf{c_{1,1}} & \mathbf{c_{1,2}} & \cdots & \mathbf{c_{1,n}} \\ \mathbf{c_{2,1}} & \mathbf{c_{2,2}} & \cdots & \mathbf{c_{2,n}} \\ \vdots & \vdots & & \vdots \\ \mathbf{c_{n,1}} & \mathbf{c_{n,2}} & \cdots & \mathbf{c_{n,n}} \end{bmatrix} \tag{59}$$

Limitations in current knowledge about damping of structural materials, or structural members built with these materials, make the described procedure difficult to apply in most practical cases. The procedure generally involves approximations based on experimentally measured damping on structures that somewhat resemble the structure under study. These procedures generally employ what is called *modal damping*. Modal damping is based on the principle that the damping matrix can be uncoupled by the vibration modes obtained from the mass and stiffness properties.

This means that matrix $[\mathbf{C}]$ when pre-multiplied by $[\Phi]^T$ and post-multiplied by $[\Phi]$ turns into a diagonal matrix:

$$[\Phi]^T[\mathbf{C}][\Phi] = [\mathbf{2}\xi_i\omega_i] \tag{60}$$

In Eq. (60) $[\mathbf{2}\xi_i\omega_i]$ is a diagonal matrix and ξ_{i_i} is the viscous damping associated with mode $\mathbf{i}$. This type of damping in which the damping matrix is uncoupled by the vibration modes obtained only from mass and stiffness matrices $[\mathbf{M}]$ and $[\mathbf{K}]$, is known as *classic damping*. However, we have to be careful that under this premise, the main property of the damping matrix is the possibility of being uncoupled by the computed modes, a mathematical property that has little relation to the physical phenomena. Having so many unknowns around, it doesn't make much sense to perform numerous computations to find a matrix $[\mathbf{C}]$ as described before; while the result would be as imprecise if we just used the values that would be obtained after performing the operation implicit in Eq. (60) Thus, the procedure generally employed consists in introducing damping to the uncoupled equation and not bothering with the computation of the damping matrix itself. This procedure is called *modal damping* and consists in using a damping value that is valid for a mode in particular. Using this procedure, uncoupled Eq. (43) turns into:

$$\ddot{\eta}_i + \mathbf{2}\xi_i\omega_i\dot{\eta}_i + \omega_i^2\eta_i = \mathbf{0} \tag{61}$$

This equation can be solved using standard techniques employed for damped single degree of freedom dynamic equations. In each uncoupled equation the coefficient of critical damping ξ_i is that corresponding to mode $\mathbf{i}$. The value to employ should be selected having in mind the type of structural material and the stress range in which the structural material would be responding when vibrating in that particular mode for the envisioned imposed displacements.

If the damping matrix $[\mathbf{C}]$ is going to be used, it is important to know the type of damping that can be uncoupled by the modes, and the following comments are relevant.

By stating that the damping matrix is a linear combination of mass $[\mathbf{M}]$ and stiffness matrix $[\mathbf{K}]$, where α and β are constants:

$$[\mathbf{C}] = \alpha[\mathbf{M}] + \beta[\mathbf{K}] \tag{62}$$

The damping matrix can be uncoupled to produce the following result:

$$[\Phi]^T[\mathbf{C}][\Phi] = \alpha[\Phi]^T[\mathbf{M}][\Phi] + \beta[\Phi]^T[\mathbf{K}][\Phi] = \alpha[\mathbf{I}] + \beta[\omega^2] = [\alpha + \beta\omega_i^2] \tag{63}$$

Where $\left[\alpha + \beta\omega_i^2\right]$ is a diagonal matrix and each one of the terms of the diagonal corresponds to $2\xi_i\omega_i$. Then the damping coefficient ξ_i for each uncoupled equation is:

$$\xi_i = \frac{\alpha}{2\omega_i} + \frac{\beta\omega_i}{2} \tag{64}$$

This type of damping is known as *Rayleigh damping*, and corresponds to a particular case of the classic damping. From Eq. (64) it is evident that damping is a function of the corresponding mode frequency, being thus different for each mode. This contradicts experimental evidence pointing to having little difference in the damping coefficients for modes belonging to the same structure. If we know experimentally obtained values for damping in two modes, say $\mathbf{r}$ and $\mathbf{s}$, it is possible to state two simultaneous equations from which we can solve for α and β:

$$\begin{Bmatrix} \xi_r \\ \xi_s \end{Bmatrix} = \frac{1}{2}\left[\begin{array}{c|c} 1/\omega_r & \omega_r \\ \hline 1/\omega_s & \omega_s \end{array}\right]\begin{Bmatrix} \alpha \\ \beta \end{Bmatrix} \tag{65}$$

If the damping coefficients of two modes are equal ($\xi = \xi_r = \xi_s$), solution of the simultaneous equations leads to:

$$\begin{Bmatrix} \alpha \\ \beta \end{Bmatrix} = \frac{2\xi}{\omega_r + \omega_s}\begin{Bmatrix} \omega_r\,\omega_s \\ 1 \end{Bmatrix} \tag{66}$$

Fig. 3 shows the relationship between damping and frequency. Cases of damping being proportional only to mass and proportional only to stiffness are also shown in the same figure.

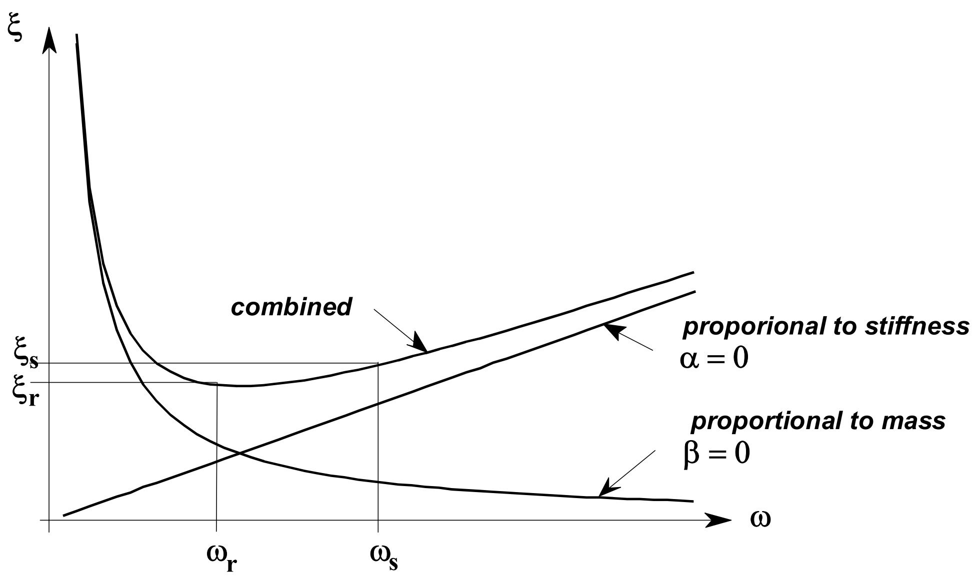


Fig. 3 - Relationship between damping and frequency for Rayleigh's damping

It is convenient to take ω_r as the value of the fundamental frequency and ω_s as the frequency corresponding to the last of the upper modes that significantly contribute to the response. This way the first mode and mode s will have exactly the same damping, and all modes in between will have somewhat smaller similar values and the modes with frequencies larger than ω_s will have larger damping values thus reducing their contribution to response.

In the literature, there are other methodologies to define a damping matrix $[\mathbf{C}]$, within the context of classical damping. Notwithstanding, the need to have a damping matrix when modal techniques are employed is not warranted since damping can be assigned to the uncoupled equation, as indicated by Eq. (61). The last statement is not necessarily true when time step-by-step solution techniques are employed as opposed to modal procedures. In the former case, the only way to introduce damping, within the domain of linear elastic response, is using a damping matrix $[\mathbf{C}]$.

The classic damping scheme should not be used in those cases in which a portion of the structure has damping significantly different from other portions. This may be the case of soil-structure evaluation, where the soil has much larger damping coefficients than the structure. In this case a technique based on employing classic damping for each portion of the structure independently to be latter combined into a single damping matrix is sometimes employed. The drawback is that this damping matrix, in some cases, is not amenable for uncoupling, thus step-by-step solution techniques must be employed.

Forced vibration

Previously we have shown that dynamic response of a multi-degree of freedom system can be solved by uncoupling the dynamic equilibrium equations. We have also shown that the free vibration solution of the response of the structure with initial conditions can be obtained as the superposition of the individual response of each one of the modes of vibration. The free vibration case corresponds to the homogeneous solution of the simultaneous differential equation system described by the dynamic equilibrium equations. Now we are interested in particular part of the solution that corresponds to the case where forces that vary in time are imposed on the degrees of freedom of the structure. We are dealing with structures whose dynamic equilibrium equations can be described, through linear algebra, in the following manner:

$$[\mathbf{M}]\{\ddot{\mathbf{x}}\}+[\mathbf{K}]\{\mathbf{x}\}=\{\mathbf{P(t)}\} \qquad (67)$$

Using the modes and frequencies of the structure obtained for free vibration; we can use the following coordinate transformation:

$$\{\mathbf{x}\}=[\Phi]\{\eta\} \qquad (68)$$

And deriving twice against time:

$$\{\ddot{x}\} = [\Phi]\{\ddot{\eta}\} \tag{69}$$

Replacing (68) and (69) in (67), and pre-multiplying by $[\Phi]^T$ we obtain:

$$\underbrace{[\Phi]^T[\mathbf{M}][\Phi]}_{[\mathbf{I}]}\{\ddot{\eta}\} + \underbrace{[\Phi]^T[\mathbf{K}][\Phi]}_{[\omega^2]}\{\eta\} = [\Phi]^T\{\mathbf{P(t)}\} \tag{70}$$

Where $[\ \mathbf{I}\ ]$ y $[\omega^2]$ are both diagonal matrices. Eq. (70) indicates that we have $\mathbf{n}$ independent differential equation of the type:

$$\ddot{\eta}_i + \omega_i^2\eta_i = \sum_{j=1}^{n}\left(\phi_j^{(i)}\ \mathbf{p}_j\mathbf{(t)}\right) \tag{71}$$

And if we use modal damping, of the type:

$$\ddot{\eta}_i + 2\xi_i\omega_i\dot{\eta} + \omega_i^2\eta_i = \sum_{j=1}^{n}\left(\phi_j^{(i)}\ \mathbf{p}_j\mathbf{(t)}\right) \tag{72}$$

Now lets study the response of forced vibration for several cases.

Harmonic forced vibration

For this case, we have an excitation represented by forces that vary in time in a constant periodic — harmonic — fashion. This could be the case of mechanical equipment that vibrates with their own frequency at different stories of a building, as shown by Fig. 4.

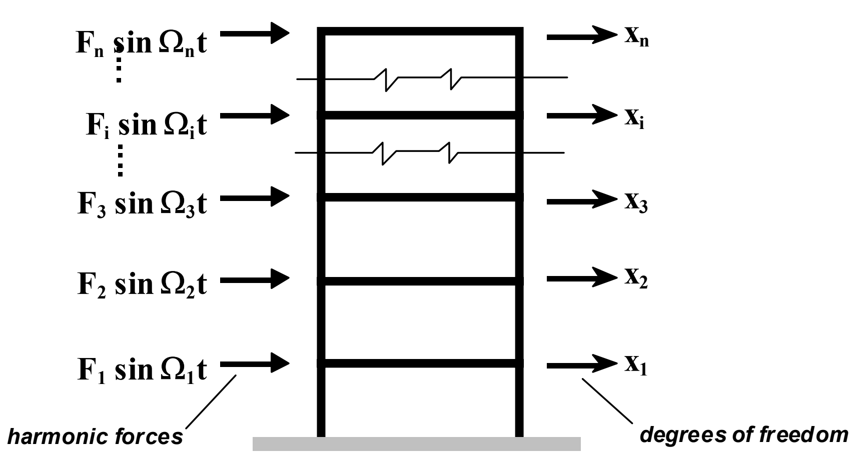


Fig. 4 - Structure subjected to several harmonic forces

The time varying forces vector has the form:

$$\left\{\mathbf{P(t)}\right\} = \left\{\mathbf{F_i\ sen\Omega_i t}\right\} = \left\{\begin{array}{c} \mathbf{F_n\ sen\Omega_n t} \\ \hline \vdots \\ \hline \mathbf{F_2\ sen\Omega_2 t} \\ \hline \mathbf{F_1\ sen\Omega_1 t} \end{array}\right\} \tag{73}$$

Where $\mathbf{F_i}$ corresponds to the amplitude — in units of force — of the harmonic force applied to $\mathbf{i}$ degree of freedom and Ω_i is the frequency — in radians per second — of the harmonic force. When the system is uncoupled as indicated by Eq. (70) $\mathbf{n}$ uncoupled equations of the type shown in Eq. (72) are obtained:

$$\ddot{\eta}_i + 2\xi_i\omega_i\dot{\eta} + \omega_i^2\eta_i = \sum_{j=1}^{n}\left(\phi_j^{(i)}\ \mathbf{F_j\ sen\Omega_j t}\right) \tag{74}$$

The solution of single degree of freedom equations under the action of a unique harmonic force, such as:

$$\ddot{\mathbf{x}} + 2\xi\omega\dot{\mathbf{x}} + \omega^2\mathbf{x} = \mathbf{F_0\ sen\Omega t} \tag{75}$$

is:

$$\mathbf{x(t)} = \Psi\,\mathbf{sen}\left(\Omega\mathbf{t} - \varphi\right) \tag{76}$$

where:

$$\Psi = \frac{\mathbf{F_0}/\omega^2}{\sqrt{\left[1-\left(\Omega/\omega\right)^2\right]^2 + \left[2\xi\left(\Omega/\omega\right)\right]^2}} \tag{77}$$

and

$$\mathbf{tan}\,\varphi = \frac{2\xi\left(\Omega/\omega\right)}{1-\left(\Omega/\omega\right)^2} \tag{78}$$

The only difference between Eq. (74) and Eq. (75) is that in the former the right side is the sum of several harmonic forces applied to the structure, affected term by term, by the appropriate values of the modal vectors. Since we are within the domain of linearly elastic response, superposition principle is valid; and response can be obtained as the superposition of each individual response. Then, the time response of the uncoupled degree of freedom η_i, is:

$$\eta_i(\mathbf{t}) = \sum_{j=1}^{n}\left(\phi_j^{(i)}\Psi_j\,\mathbf{sen}\left(\Omega_j\mathbf{t} - \varphi_j\right)\right) \tag{79}$$

where:

$$\Psi_j = \frac{F_j / \omega_i^2}{\sqrt{\left[1-\left(\Omega_j/\omega_i\right)^2\right]^2 + \left[2\xi_i\left(\Omega_j/\omega_i\right)\right]^2}} \tag{80}$$

and

$$\tan\varphi_j = \frac{2\xi_i\left(\Omega_j/\omega_i\right)}{1-\left(\Omega_j/\omega_i\right)^2} \tag{81}$$

In Eqs. (80) and (81) **i** corresponds to the sub index of the uncoupled equation and **j** to the sub index of the degree of freedom where the harmonic force is applied. Once the values of all η_i variables are obtained, the displacements of the degrees of freedom used to state equilibrium may be obtained from:

$$\{x\} = [\Phi]\{\eta\} \tag{82}$$

The response obtained from this procedure is the particular solution of the differential equations and correspond to the steady state where the effect of the initial conditions is not present or has disappeared due to damping. Base of this, the definition of the time of initiation of the excitation loses importance, and the given enough time the responses will combine in their maximum values. Thus, the maximum possible response can be obtained from the sum of the absolute values of the maximum individual responses:

$$\left(\eta_i\right)_{max} = \sum_{j=1}^{n}\left|\phi_j^{(i)}\Psi_j\right| \tag{83}$$

Transient forced vibration

For an arbitrary excitation, the solution of the uncoupled equations is performed using the same techniques than those employed for single degrees of freedom systems. One way to perform this solution is using the convolution integral, also known as Duhamel integral:

$$\eta_i(t) = \frac{1}{\omega_i\sqrt{1-\xi_i^2}}\int_0^t \overline{P}_i(\tau)\, e^{-\xi_i\omega_i(t-\tau)}\, \text{sen}\left\{\sqrt{1-\xi_i^2}\,\omega_i(t-\tau)\right\} d\tau \tag{84}$$

where:

$$\overline{P}_i(t) = \sum_{j=1}^{n}\left(\phi_j^{(i)} p_j(t)\right) \tag{85}$$

In the two last equations, sub index **i** refers to the uncoupled equation that corresponds to mode **i** and sub index **j** to the degree of freedom, of those employed in stating dynamic equilibrium of the structure.

Alternatively to the use of Duhamel's integral, any of the numerical method used for solving single degree of freedom systems can be used.

Example 4

The building shown in Fig. 5 is subjected to an explosion. The air pressure wave caused by the explosion varies in the form shown in Fig. 5(b). We are interested in obtaining the response of the structure in the short direction, as shown in the figure. Damping of the structure, for the displacement amplitude expected, is estimated to be **ξ = 2%** *of critical. All girders of the frames have width* **b = 0.40 m** *and depth* **h = 0.50 m**. *All columns are square with a section side dimension of* **h = 0.40 m**. *The modulus of elasticity of the structure is* **E = 25 GPa**. *The building has a mass per unit area of* **1000 kg/m^2**.

The explosion occurred far away, therefore we can assume that the pressure applied to the building doesn't vary with height and is applied uniformly to the building façade. The tributary area for application of the pressure at the top story is **10m · 1.5 m = 15 m^2** *and for the other floors* **10m · 3 m = 30 m^2**.

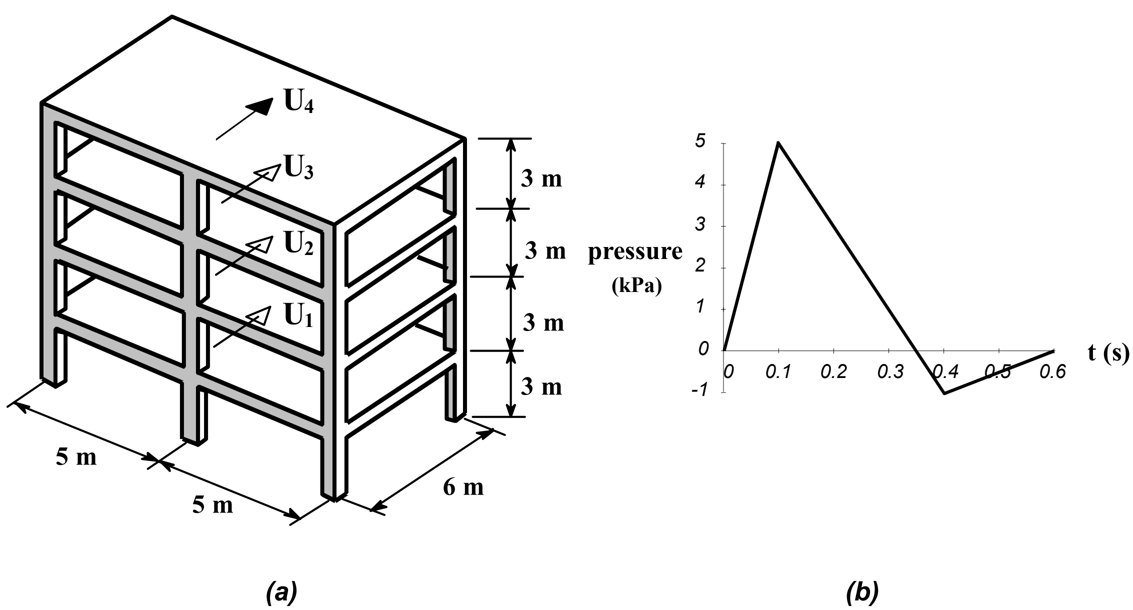


Fig. 5 - Example 4

We must first find the stiffness characteristics of the building in the short direction. Since the three frames have the same member dimensions and a rigid diaphragm effect is envisioned for the structure, the lateral stiffness properties for one frame are obtained and simply will be multiplied by three to obtain the total lateral building stiffness in the short direction. The frame stiffness is obtained using matrix analysis considering it a plane frame with three degrees of freedom per joint — a horizontal displacement, a vertical displacement and a rotation around an axis perpendicular to the plane of the frame. Once the stiffness matrix is obtained for these degrees of freedom, the rigid diaphragm condition is imposed by making all lateral displacements in the same story equal. Then all vertical displacements are condensed, leaving only the degrees of freedom corresponding to the lateral displacements of the frame expressed in a 4 x 4 stiffness matrix. After multiplying this matrix by three to take into account the effect of all three frames, the following stiffness matrix for the building in the short direction in kN/m was obtained:

$$\left[\mathbf{K_E}\right] = 10^3 \times \left[\begin{array}{r|r|r|r} 76.869 & -99.691 & 25.583 & -3.4747 \\ \hline -99.691 & 209.14 & -136.02 & 31.108 \\ \hline 25.583 & -136.02 & 221.76 & -142.11 \\ \hline -3.4747 & 31.108 & -142.11 & 252.10 \end{array}\right] \begin{array}{l} \downarrow \mathbf{dof} \\ \mathbf{U_4} \\ \mathbf{U_3} \\ \mathbf{U_2} \\ \mathbf{U_1} \end{array}$$

Each floor slab has an area of **10 m · 6 m = 60 m²**. Therefore, the translational mass per story is **m = 60 m²· 1000 kg/m² = 60 Mg**. The mass matrix for the building is then:

$$\left[\mathbf{M}\right] = \left[\begin{array}{r|r|r|r} 60 & 0 & 0 & 0 \\ \hline 0 & 60 & 0 & 0 \\ \hline 0 & 0 & 60 & 0 \\ \hline 0 & 0 & 0 & 60 \end{array}\right] \begin{array}{l} \downarrow \mathbf{dof} \\ \mathbf{U_4} \\ \mathbf{U_3} \\ \mathbf{U_2} \\ \mathbf{U_1} \end{array}$$

We can state the dynamic equilibrium equations as:

$$\left[\begin{array}{r|r|r|r} 60 & 0 & 0 & 0 \\ \hline 0 & 60 & 0 & 0 \\ \hline 0 & 0 & 60 & 0 \\ \hline 0 & 0 & 0 & 60 \end{array}\right] \left\{\begin{array}{c} \ddot{U}_4 \\ \hline \ddot{U}_3 \\ \hline \ddot{U}_2 \\ \hline \ddot{U}_1 \end{array}\right\} + 10^3 \times \left[\begin{array}{r|r|r|r} 76.869 & -99.691 & 25.583 & -3.4747 \\ \hline -99.691 & 209.14 & -136.02 & 31.108 \\ \hline 25.583 & -136.02 & 221.76 & -142.11 \\ \hline -3.4747 & 31.108 & -142.11 & 252.10 \end{array}\right] \left\{\begin{array}{c} U_4 \\ \hline U_3 \\ \hline U_2 \\ \hline U_1 \end{array}\right\} = \left\{\begin{array}{c} 15 \\ \hline 30 \\ \hline 30 \\ \hline 30 \end{array}\right\} q(t)$$

The right side corresponds to the façade tributary area in square meters assignable to each story and **q(t)** is the explosion caused pressure described in Fig. 5(b) The tributary area in square meters multiplied by a pressure in kPa, give a force in kN, consistent with units from the product of masses in Mg by accelerations in m/s^2, thus resulting in forces in kN and stiffnesses in kN/m per m of displacement, giving forces also in kN.

By solving the eigenvalues problem stated in the dynamic equilibrium equation, the following frequencies and periods are obtained:

Mode	ω^2 (rad/s)²	ω (rad/s)	**f** (Hertz)	**T** (s)
1	115.22	10.73	1.708	0.59
2	1176.5	34.30	5.458	0.18
3	3820.2	61.80	9.836	0.10
4	7552.6	86.90	13.83	0.072

The corresponding vibration modes are:

$$[\Phi] = \begin{bmatrix} \mathbf{0.089374} & \mathbf{-0.074828} & \mathbf{0.050226} & \mathbf{-0.023604} \\ \mathbf{0.075047} & \mathbf{0.014665} & \mathbf{-0.081059} & \mathbf{0.065184} \\ \mathbf{0.050937} & \mathbf{0.083904} & \mathbf{-0.004977} & \mathbf{-0.083710} \\ \mathbf{0.021268} & \mathbf{0.061745} & \mathbf{0.086883} & \mathbf{0.069665} \end{bmatrix}$$

(These modes are normalized to meet $[\Phi]^T[M][\Phi] = [I]$, therefore are orthonormal)

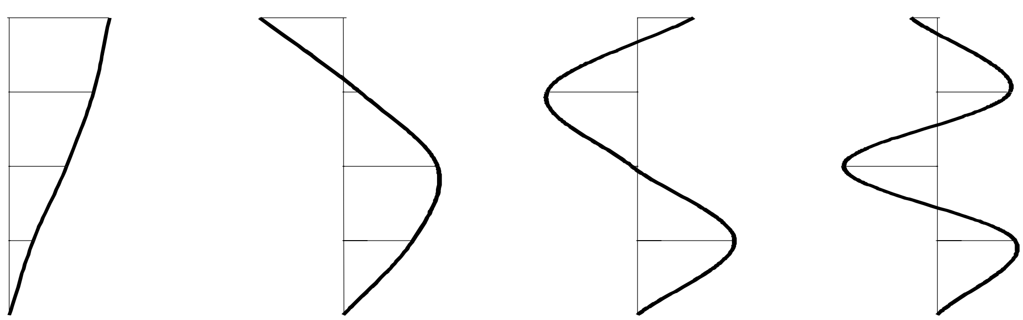


Fig. 6 - Example 4 - Vibration modes and periods of the structure

Using the following coordinate transformation equations in the dynamic equilibrium equations:

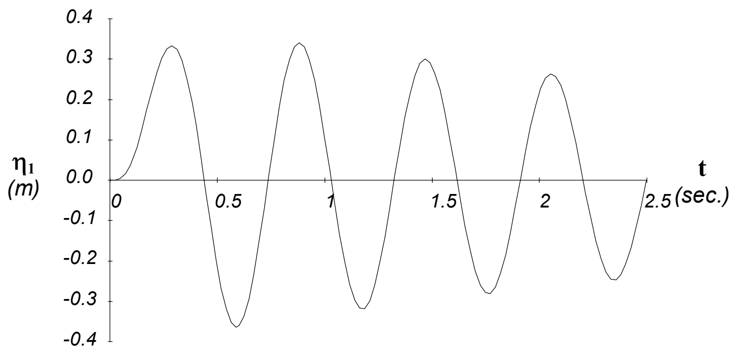
$$\{U\} = [\Phi]\{\eta\} \quad \text{and} \quad \{\ddot{U}\} = [\Phi]\{\ddot{\eta}\}$$

And pre-multiplying both sides by $[\Phi]^T$, the following uncoupled equations are obtained:

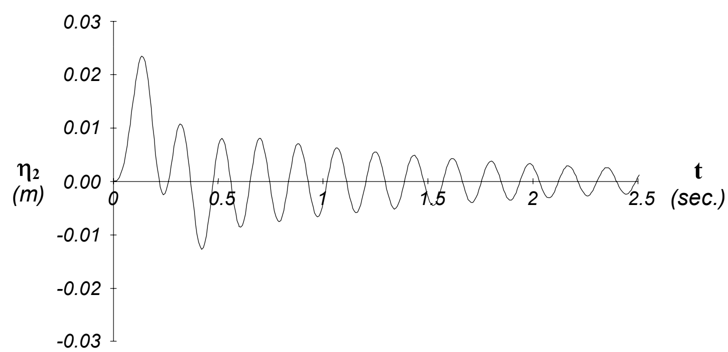
$$\ddot{\eta}_1 + 2\xi_1\omega_1\dot{\eta}_1 + \omega_1^2\eta_1 = 5.7582\,q(t)$$
$$\ddot{\eta}_2 + 2\xi_2\omega_2\dot{\eta}_2 + \omega_2^2\eta_2 = 3.6870\,q(t)$$
$$\ddot{\eta}_3 + 2\xi_3\omega_3\dot{\eta}_3 + \omega_3^2\eta_3 = 0.77879\,q(t)$$
$$\ddot{\eta}_4 + 2\xi_4\omega_4\dot{\eta}_4 + \omega_4^2\eta_4 = 1.1801\,q(t)$$

In these four equations $\xi_i = 0.02$. The response of each of the uncoupled equations was obtained employing Newmark's Beta method. The first 2.5 of response are shown in Fig. 7:

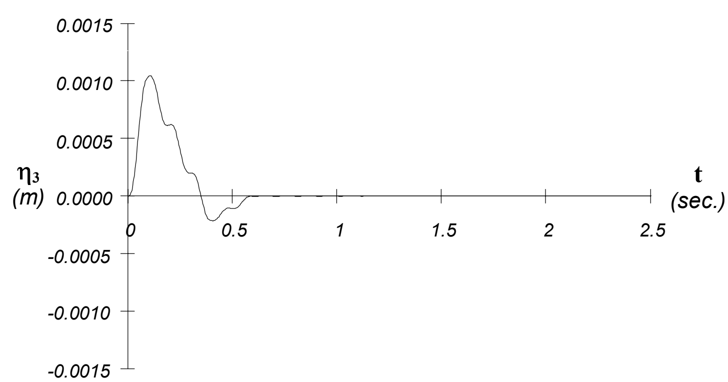
η_1(t) - (T$_1$ = *0.59 s)*



η_2(t) - (T$_2$ = *0.18 s)*



η_3(t) - (T$_3$ = *0.10 s)*:



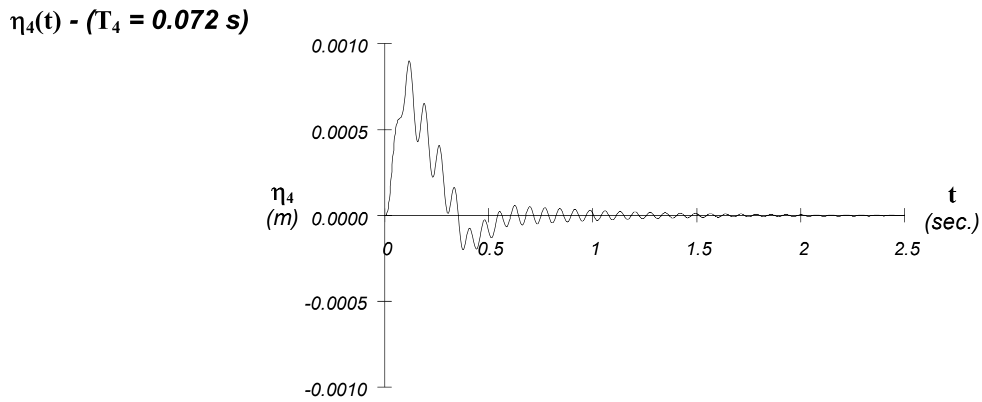


Fig. 7 - Example 4 - Response in time for the uncoupled degrees of freedom

The response at some instants are presented in the following table:

t (s)	η_1 (m)	η_2 (m)	η_3 (m)	η_4 (m)
0.0000	0.000000	0.000000	0.000000	0.000000
0.0234	0.000607	0.000392	0.000074	0.000097
0.0468	0.004810	0.002748	0.000425	0.000419
0.0702	0.015963	0.007780	0.000846	0.000563
0.0937	0.036972	0.014603	0.001025	0.000656
0.1000	0.044673	0.016511	0.001037	0.000724
0.1234	0.080485	0.022223	0.001025	0.000891
0.1468	0.125033	0.022946	0.000840	0.000548
0.1702	0.174003	0.017556	0.000657	0.000481
0.1937	0.222849	0.008559	0.000613	0.000648
0.2171	0.267077	0.000513	0.000612	0.000357
0.2405	0.302516	-0.002488	0.000491	0.000249
0.2639	0.325557	0.000139	0.000300	0.000406
0.2873	0.333435	0.005794	0.000206	0.000170
0.3107	0.324369	0.010167	0.000200	0.000022
0.4000	0.135337	-0.009842	-0.000213	-0.000097

The structure displacements are obtained from:

$$\{\mathbf{U}\} = [\Phi]\{\eta\}$$

For example, for instant **t=0.2873 s**, displacements in m for each mode and total values are:

$$\begin{Bmatrix} U_4 \\ U_3 \\ U_2 \\ U_1 \end{Bmatrix} = \begin{bmatrix} 0.089374 & -0.074828 & 0.050226 & -0.023604 \\ 0.075047 & 0.014665 & -0.081059 & 0.065184 \\ 0.050937 & 0.083904 & -0.004977 & -0.083710 \\ 0.021268 & 0.061745 & 0.086883 & 0.069665 \end{bmatrix} \begin{Bmatrix} 0.333435 \\ 0.005794 \\ 0.000206 \\ 0.000170 \end{Bmatrix}$$

$$= \begin{Bmatrix} 0.089374 \\ 0.075047 \\ 0.050937 \\ 0.021268 \end{Bmatrix} \times 0.333435 + \begin{Bmatrix} -0.074828 \\ 0.014665 \\ 0.083904 \\ 0.061745 \end{Bmatrix} \times 0.005794 + \begin{Bmatrix} 0.050226 \\ -0.081059 \\ -0.004977 \\ 0.086883 \end{Bmatrix} \times 0.000206 + \begin{Bmatrix} -0.023604 \\ 0.065184 \\ -0.083710 \\ 0.069665 \end{Bmatrix} \times 0.000170$$

$$= \begin{Bmatrix} 0.029800 \\ 0.025023 \\ 0.016984 \\ 0.007992 \end{Bmatrix} + \begin{Bmatrix} -0.000434 \\ 0.000085 \\ 0.000486 \\ 0.000358 \end{Bmatrix} + \begin{Bmatrix} 0.0000103 \\ -0.0000167 \\ -0.0000010 \\ 0.0000179 \end{Bmatrix} + \begin{Bmatrix} -0.0000040 \\ 0.0000111 \\ -0.0000142 \\ 0.0000118 \end{Bmatrix}$$

$$= \begin{Bmatrix} 0.029373 \\ 0.025103 \\ 0.017455 \\ 0.007479 \end{Bmatrix}$$

To obtain the forces caused by the explosion at the same instant for all the structure, the structure stiffness matrix is multiplied by the displacements obtained:

$$\{F\} = [K_E]\{U\}$$

This operation can be made for each mode independently in order to obtain displacements of the structure for each mode:

$$\{F^{mod}\} = [K_E]\left[U^{(1)} \mid U^{(2)} \mid U^{(3)} \mid U^{(4)}\right] = \left[F^{(1)} \mid F^{(2)} \mid F^{(3)} \mid F^{(4)}\right]$$

Now, for the modal displacements at instant **t=0.2873 s***:*

$$\{U^{mod}\} = \{U^{(1)} \mid U^{(2)} \mid U^{(3)} \mid U^{(4)}\} = \begin{bmatrix} 0.029800 & -0.000434 & 0.0000103 & -0.0000040 \\ 0.025023 & 0.000085 & -0.0000167 & 0.0000111 \\ 0.016984 & 0.000486 & -0.0000010 & -0.0000142 \\ 0.007992 & 0.000358 & 0.0000179 & 0.0000118 \end{bmatrix}$$

The contribution to the applied force caused by each mode, in kN, at instant **t=0.2873 s***, is:*

$$\left\{\mathbf{F}^{\mathrm{mod}}\right\}=\left[\mathbf{K}_{\mathrm{E}}\right]\left\{\mathbf{U}^{\mathrm{mod}}\right\}=\left[\mathbf{F}^{(1)}\mid\mathbf{F}^{(2)}\mid\mathbf{F}^{(3)}\mid\mathbf{F}^{(4)}\right]=\left[\begin{array}{r|r|r|r}\mathbf{206.02} & \mathbf{-39.60} & \mathbf{2.37} & \mathbf{-1.82}\\ \hline \mathbf{172.99} & \mathbf{6.00} & \mathbf{-3.83} & \mathbf{5.02}\\ \hline \mathbf{117.41} & \mathbf{34.32} & \mathbf{-0.24} & \mathbf{-6.45}\\ \hline \mathbf{49.03} & \mathbf{25.25} & \mathbf{4.10} & \mathbf{5.37}\end{array}\right]$$

And total forces in kN, for instant **t=0.2873 s**, *are:*

$$\left\{\mathbf{F}\right\}=\left[\mathbf{K}_{\mathrm{E}}\right]\left\{\mathbf{U}\right\}=\left\{\begin{array}{r}\mathbf{175.97}\\ \hline \mathbf{180.18}\\ \hline \mathbf{145.05}\\ \hline \mathbf{83.75}\end{array}\right\}$$

■

Base excitation

Now lets study the base excitation of a multi-degree of freedom system, such as earthquake ground motions. Base dynamic excitation equilibrium equations have the following form:

$$\left[\mathbf{M}\right]\left\{\ddot{\mathbf{U}}\right\}+\left[\mathbf{K}\right]\left\{\mathbf{U}\right\}=-\left[\mathbf{M}\right]\left[\gamma\right]\left\{\ddot{\mathbf{x}}_{\mathbf{o}}\right\} \tag{86}$$

Matrix $[\gamma]$ connects the direction of the degrees of freedom expressed in the equilibrium equations with the appropriate component of the accelerogram. Its form depends on the number of accelerogram components employed (one, two, or three) in vector $\left\{\ddot{\mathbf{x}}_{\mathbf{0}}\right\}$. $[\gamma]$ has as many columns as components of the accelerogram are employed.

We use the same procedure to uncouple the dynamic equilibrium equations by applying the following coordinate transformation:

$$\left\{\mathbf{U}\right\}=\left[\Phi\right]\left\{\eta\right\} \tag{87}$$

And deriving twice against time:

$$\left\{\ddot{\mathbf{U}}\right\}=\left[\Phi\right]\left\{\ddot{\eta}\right\} \tag{88}$$

Replacing Eqs. (87) and (88) in (86) and pre-multiplying by $[\Phi]^{\mathrm{T}}$ we obtain:

$$\underbrace{\left[\Phi\right]^{\mathrm{T}}\left[\mathbf{M}\right]\left[\Phi\right]}_{\left[\mathbf{I}\right]}\left\{\ddot{\eta}\right\}+\underbrace{\left[\Phi\right]^{\mathrm{T}}\left[\mathbf{K}\right]\left[\Phi\right]}_{\left[\omega^2\right]}\left\{\eta\right\}=-\left[\Phi\right]^{\mathrm{T}}\left[\mathbf{M}\right]\left[\gamma\right]\left\{\ddot{\mathbf{x}}_{\mathbf{o}}\right\} \tag{89}$$

This means having $\mathbf{n}$ independent single degree of freedom equation of type:

$$\ddot{\eta}_i + \omega_i^2 \eta_i = -\{\alpha_i\}\{\ddot{x}_o\} \tag{90}$$

And if modal damping is used:

$$\ddot{\eta}_i + 2\xi_i\omega_i\dot{\eta} + \omega_i^2 \eta_i = -\{\alpha_i\}\{\ddot{x}_o\} \tag{91}$$

In Eqs. (90) and (91) $\{\alpha_i\}$ is called the *participation coefficient* and corresponds to row $\mathbf{i}$ of matrix $[\alpha]$ obtained from:

$$[\alpha] = [\Phi]^T [\mathbf{M}][\gamma] \tag{92}$$

Solution of Eqs. (90) and (91) can be performed using a suitable numerical method such as Newmark's Beta method. Once the values of $\{\eta(\mathbf{t})\}$ are known, for any time instant $\mathbf{t}$, using Eq. (87) the displacements of the structure for that instant may be computed. It should be noted that Eq. (87) performs the superposition of the response of all modes directly.

$$\begin{aligned}\{\mathbf{U}\} &= [\Phi]\{\eta\} = \sum_{i=1}^{n}\{\phi^{(i)}\}\eta_i(\mathbf{t}) \\ &= \{\phi^{(1)}\}\eta_1(\mathbf{t}) + \{\phi^{(2)}\}\eta_2(\mathbf{t}) + \cdots + \{\phi^{(n)}\}\eta_n(\mathbf{t}) \\ &= \{\mathbf{U}^{(1)}\} + \{\mathbf{U}^{(2)}\} + \cdots + \{\mathbf{U}^{(n)}\}\end{aligned} \tag{93}$$

The forces imposed by the ground motion for each mode can be obtained by multiplying the displacements caused by each mode by the stiffness matrix of the structure:

$$\{\mathbf{F}^{(i)}\} = [\mathbf{K}]\{\mathbf{U}^{(i)}\} \tag{94}$$

Defining:

$$\{\mathbf{1}\} = \begin{Bmatrix} \mathbf{1} \\ - \\ \mathbf{1} \\ - \\ \vdots \\ - \\ \mathbf{1} \end{Bmatrix} \tag{95}$$

and

$$\{\mathbf{h}\} = \begin{Bmatrix} \mathbf{h_n} \\ \hline \mathbf{h_i} \\ \hline \vdots \\ \hline \mathbf{h_1} \end{Bmatrix} \tag{96}$$

where $\mathbf{h_i}$ is the height of story $\mathbf{i}$ measured from the base of the structure.

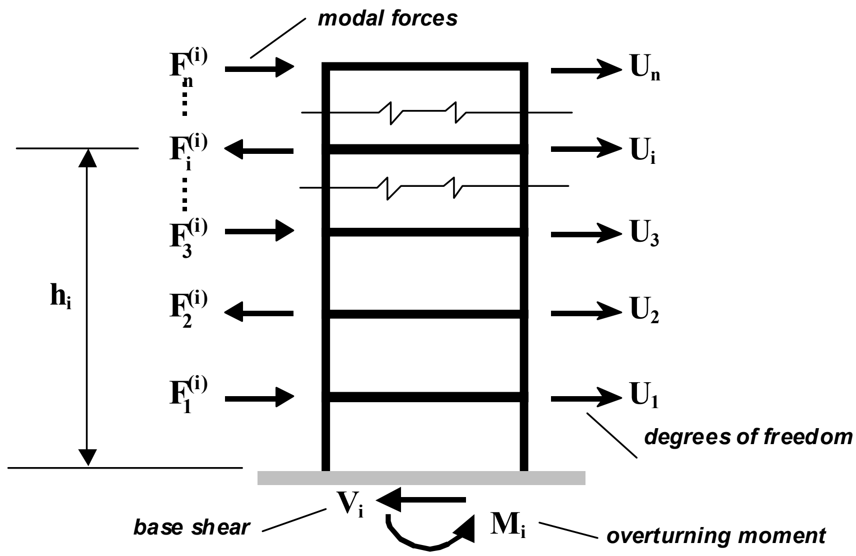


Fig. 8 - Modal forces (Mode i)

The base shear caused by mode $\mathbf{i}$ at instant $\mathbf{t}$, is

$$\mathbf{V_i} = \{\mathbf{1}\}^T \{\mathbf{F}^{(i)}\} \tag{97}$$

The overturning moment of mode $\mathbf{i}$ at instant $\mathbf{t}$, is:

$$\mathbf{M_i} = \{\mathbf{h}\}^T \{\mathbf{F}^{(i)}\} \tag{98}$$

Now, using the definition of matrix $[\alpha]$ given in Eq. (92) and pre-multiplying it by $[\Phi]^T$, we obtain:

$$[\Phi]^T[\alpha] = [\Phi]^T[\Phi]^T[\mathbf{M}][\gamma] \tag{99}$$

Using the principle that $([\mathbf{A}][\mathbf{B}])^T = [\mathbf{B}]^T[\mathbf{A}]^T$ to $[\Phi]^T[\mathbf{M}]$, we obtain $[\mathbf{M}]^T[\Phi] = [\mathbf{M}][\Phi]$, since $[\mathbf{M}]$ is symmetric. With this Eq. (99) converts into:

$$[\Phi]^T[\alpha] = [\Phi]^T[\mathbf{M}][\Phi][\gamma] = [\gamma] \tag{100}$$

The total mass of the structure for any principal direction of the degrees of freedom corresponds to the sum of all masses that acts in that direction. The influence of each individual mass is expressed through matrix $[\gamma]$, then:

$$\left[\mathbf{M}_{tot}\right] = \left[\gamma\right]^{T}\left[\mathbf{M}\right]\left[\gamma\right] \tag{101}$$

Now using Eq. (100) to replace $[\gamma]$ in Eq. (101) the following results are obtained:

$$\left[\mathbf{M}_{tot}\right] = \left(\left[\Phi\right]^{T}\left[\alpha\right]\right)^{T}\left[\mathbf{M}\right]\left[\Phi\right]^{T}\left[\alpha\right] = \left[\alpha\right]^{T}\left[\Phi\right]\left[\mathbf{M}\right]\left[\Phi\right]^{T}\left[\alpha\right] \tag{102}$$

and

$$\left[\mathbf{M}_{tot}\right] = \left[\alpha\right]^{T}\left[\mathbf{I}\right]\left[\alpha\right] = \left[\sum \alpha_i^2\right] \tag{103}$$

This means that the total mass that acts in a principal direction is the sum of the square of the modal participation coefficients, α_i, in that direction. The value of α^2 for each mode is called *effective or active modal mass* and can be interpreted as the fraction of the total mass that is activated by the mode when vibrating due to base excitation. This concept is used for defining the minimum number of modes necessary to describe the response in systems with many degrees of freedom and where the contribution of the upper modes to the response is not significant.

In those cases where the normalization procedure employed for the modes doesn't lead to orthonormal modes and $[\Phi]^T[\mathbf{M}][\Phi] = [\,\mathbf{I}\,]$ is not complied with, participation coefficients must be obtained from the following equation:

$$\alpha_i = \frac{\left\{\phi^{(i)}\right\}^{T}\left[\mathbf{M}\right]\left\{\gamma\right\}}{\left\{\phi^{(i)}\right\}^{T}\left[\mathbf{M}\right]\left\{\phi^{(i)}\right\}} = \frac{\sum\limits_{j=1}^{n}\left(\phi_j^{(i)}\mathbf{m}_j\right)}{\sum\limits_{j=1}^{n}\mathbf{m}_j\left(\phi_j^{(i)}\right)^2} \tag{104}$$

And the mode effective mass should be obtained, in this case, using:

$$\mathbf{m}_{ef}^{(i)} = \frac{\left(\left\{\phi^{(i)}\right\}^{T}\left[\mathbf{M}\right]\left\{\gamma\right\}\right)^2}{\left\{\phi^{(i)}\right\}^{T}\left[\mathbf{M}\right]\left\{\phi^{(i)}\right\}} = \frac{\left[\sum\limits_{j=1}^{n}\left(\phi_j^{(i)}\mathbf{m}_j\right)\right]^2}{\sum\limits_{j=1}^{n}\mathbf{m}_j\left(\phi_j^{(i)}\right)^2} \tag{105}$$

Now lets look at the response of planar systems to earthquake ground motions at the base of the structure. In this case matrix $[\gamma]$ corresponds to a vector $\{\mathbf{1}\}$ having ones in all its terms since all the lateral degrees of freedom of the structure are collinear with the accelerogram acting at the base of the structure.

Example 5

Fig. 9 shows a building that is part of an industrial facility. We want to study the response of the building to the N-S component of the recorded accelerations at El Centro, California, in Mayo 18 of 1940. We are interested in the response in the direction shown in the figure. Damping for the system was estimated in ξ **= 5%** *of critical. All girders of the structure have width* **b = 0.40 m** *and depth* **h = 0.50 m**. *All columns have square section with a cross section dimension* **h = 0.50 m**. *The material of the structure has a modulus of elasticity* **E = 25 GPa**. *The self weight of structure plus additional dead load is* **780 kg/m^2** *and the industrial machinery, which is firmly connected to the building slabs, increases the mass per unit area by* **1000 kg/m^2**, *for a total mass per unit area of* **1780 kg/m^2**.

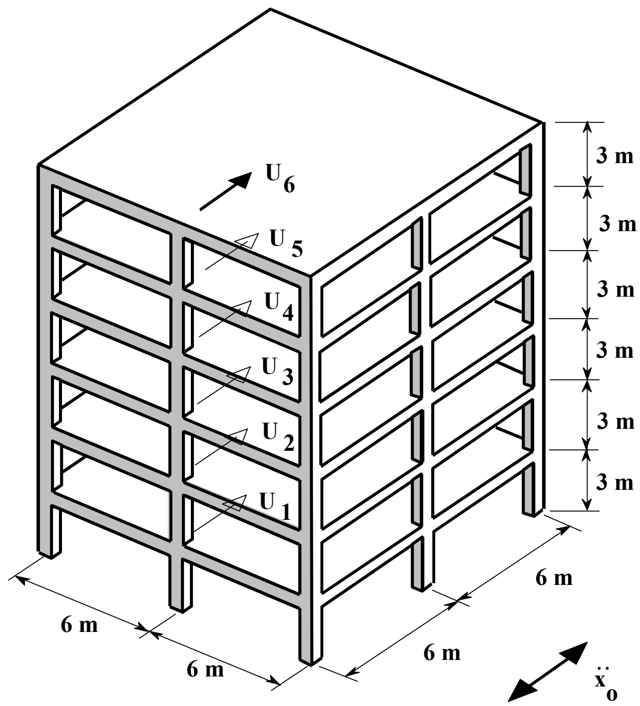


Fig. 9 - Example 5

The first step is to obtain the stiffness properties of the structure in the direction of the ground acceleration. A rigid diaphragm scheme is employed; therefore, the frames in that direction will have compatible lateral displacements. Since the three frames in that direction have the same properties, once the lateral stiffness of one frame is obtained it should be multiplied by three to obtain the lateral stiffness of the whole structure in the direction of interest. The frame stiffness matrix is modified to eliminate any axial deformations of the girders (to comply with the rigid diaphragm condition), and the vertical deformations and joint rotations are condensed. After performing all these operations, the lateral-load *stiffness matrix of the structure in the direction of the ground acceleration in kN/m is:*

$$
\left[\mathbf{K_E}\right] = 10^3 \times \begin{array}{c} \\ \left[\begin{array}{r|r|r|r|r|r} 216.76 & -306.77 & 105.49 & -19.561 & 4.2822 & -0.51088 \\ \hline -306.77 & 668.24 & -475.14 & 137.94 & -29.375 & 5.3857 \\ \hline 105.49 & -475.14 & 731.37 & -493.23 & 159.60 & -29.327 \\ \hline -19.561 & 137.94 & -493.23 & 749.02 & -494.47 & 145.71 \\ \hline 4.2822 & -29.375 & 159.60 & -494.47 & 738.11 & -515.90 \\ \hline -0.51088 & 5.3857 & -29.327 & 145.71 & -515.90 & 889.94 \end{array}\right] \end{array} \begin{array}{l} \downarrow \textbf{dof} \\ U_6 \\ U_5 \\ U_4 \\ U_3 \\ U_2 \\ U_1 \end{array}
$$

The area of each floor slab is **12 m · 12 m = 144 m²**. *The total translational mass of each story is* **m = 144 m²· 1780 kg/m² = 256 Mg**. *The mass matrix of the buildings is:*

$$
\left[\mathbf{M}\right] = \left[\begin{array}{r|r|r|r|r|r} 256 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 256 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 256 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 256 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 256 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 256 \end{array}\right] \begin{array}{l} \downarrow \textbf{dof} \\ U_6 \\ U_5 \\ U_4 \\ U_3 \\ U_2 \\ U_1 \end{array}
$$

Matrix **[γ]** *is in this case a single column vector having one in all rows, because all the lateral degrees of freedom of the structure are parallel to the ground motion acceleration. The dynamic equilibrium equations are:*

$$
\left[\begin{array}{r|r|r|r|r|r} 256 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 256 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 256 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 256 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 256 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 256 \end{array}\right] \left\{\begin{array}{c} \ddot{U}_6 \\ \hline \ddot{U}_5 \\ \hline \ddot{U}_4 \\ \hline \ddot{U}_3 \\ \hline \ddot{U}_2 \\ \hline \ddot{U}_1 \end{array}\right\}
$$

$$
+10^3 \times \left[\begin{array}{r|r|r|r|r|r} 216.76 & -306.77 & 105.49 & -19.561 & 4.2822 & -0.51088 \\ \hline -306.77 & 668.24 & -475.14 & 137.94 & -29.375 & 5.3857 \\ \hline 105.49 & -475.14 & 731.37 & -493.23 & 159.60 & -29.327 \\ \hline -19.561 & 137.94 & -493.23 & 749.02 & -494.47 & 145.71 \\ \hline 4.2822 & -29.375 & 159.60 & -494.47 & 738.11 & -515.90 \\ \hline -0.51088 & 5.3857 & -29.327 & 145.71 & -515.90 & 889.94 \end{array}\right] \left\{\begin{array}{c} U_6 \\ \hline U_5 \\ \hline U_4 \\ \hline U_3 \\ \hline U_2 \\ \hline U_1 \end{array}\right\} = -\left[\mathbf{M}\right] \left\{\begin{array}{c} 1 \\ \hline 1 \\ \hline 1 \\ \hline 1 \\ \hline 1 \\ \hline 1 \end{array}\right\} \ddot{x}_0
$$

After solving the eigenvalues problem for this system, we find:

Mode	ω^2 (rad/s)²	ω (rad/s)	f (Hertz)	T (s)
1	29.108	5.39	0.859	1.16
2	301.81	17.4	2.76	0.36
3	973.78	31.2	4.97	0.20
4	2494.3	49.9	7.95	0.13
5	4686.5	68.5	10.9	0.092
6	7113.8	84.3	13.4	0.075

The corresponding vibration modes are:

$$[\Phi] = \begin{bmatrix} 0.036721 & -0.032775 & 0.029168 & -0.020667 & 0.013049 & -0.005955 \\ 0.033690 & -0.011592 & -0.014245 & 0.032483 & -0.032188 & 0.018512 \\ 0.028524 & 0.014524 & -0.034529 & 0.005317 & 0.028533 & -0.029103 \\ 0.020961 & 0.033322 & -0.005049 & -0.034504 & -0.003317 & 0.033609 \\ 0.012243 & 0.033525 & 0.031633 & 0.006893 & -0.024392 & -0.031454 \\ 0.004460 & 0.015888 & 0.025184 & 0.034025 & 0.035774 & 0.023711 \end{bmatrix}$$

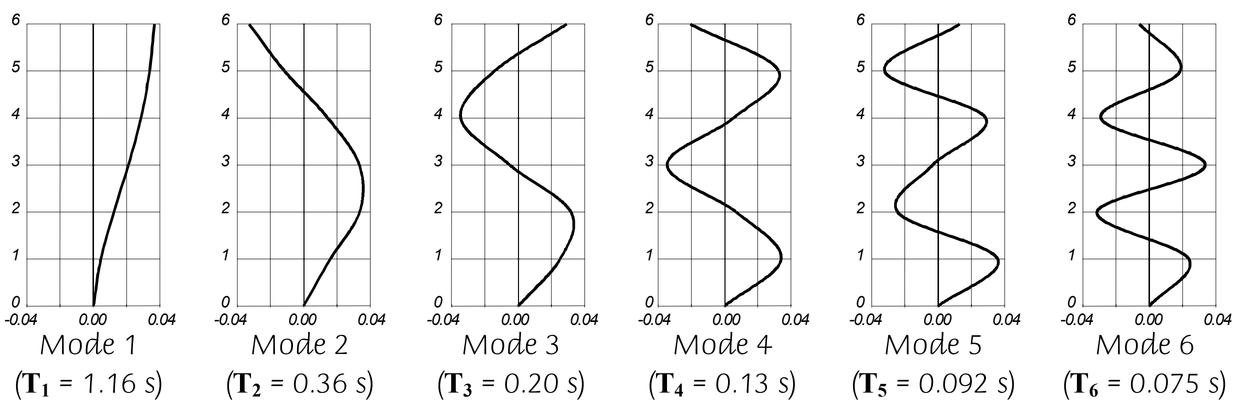


Fig. 10 - Example 5 - Structure vibration periods and modes

The modal participation factors are obtained from:

$$\{\alpha\} = [\Phi]^T [\mathbf{M}][\gamma] = \begin{Bmatrix} 34.970 \\ 13.540 \\ 8.2331 \\ 6.0279 \\ 4.4695 \\ 2.3861 \end{Bmatrix}$$

The total effective mass is computed as α_i^2

Mode	α_i	α_i^2	$\%M_{tot}$	$\%M_{tot}$ accumulated
1	34.970	1222.901	79.62%	79.62%
2	13.540	183.332	11.93%	91.55%
3	8.2331	67.784	4.41%	95.96%
4	6.0279	36.336	2.37%	98.33%
5	4.4695	19.976	1.30%	99.63%
6	2.3861	5.693	0.37%	100.00%

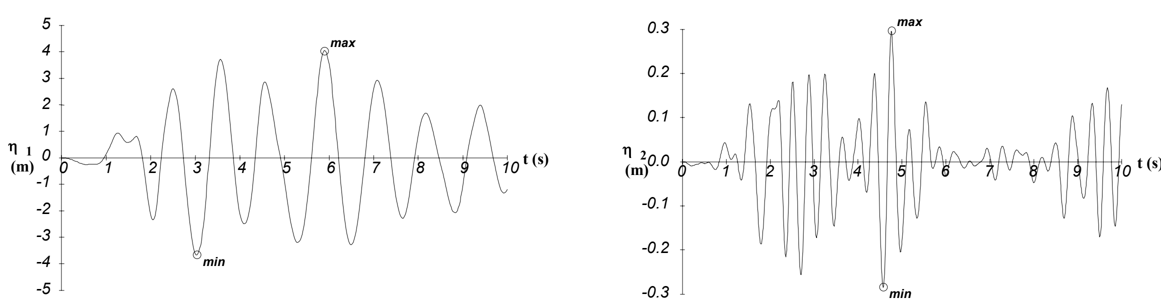
Now we modify the dynamic equilibrium equations by pre-multiplying by $[\Phi]^T$ and using the following coordinate transformations:

$$\{U\} = [\Phi]\{\eta\} \quad \text{and} \quad \{\ddot{U}\} = [\Phi]\{\ddot{\eta}\}$$

The uncoupled vibration equations are:

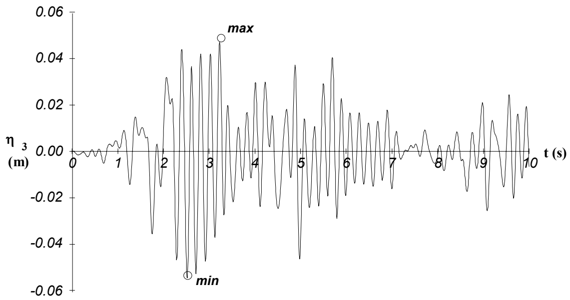
$$\begin{aligned}
\ddot{\eta}_1 + 2\xi_1\omega_1\dot{\eta}_1 + \omega_1^2\eta_1 &= -34.970\,\ddot{x}_0 \\
\ddot{\eta}_2 + 2\xi_2\omega_2\dot{\eta}_2 + \omega_2^2\eta_2 &= -13.540\,\ddot{x}_0 \\
\ddot{\eta}_3 + 2\xi_3\omega_3\dot{\eta}_3 + \omega_3^2\eta_3 &= -8.2331\,\ddot{x}_0 \\
\ddot{\eta}_4 + 2\xi_4\omega_4\dot{\eta}_4 + \omega_4^2\eta_4 &= -6.0279\,\ddot{x}_0 \\
\ddot{\eta}_5 + 2\xi_5\omega_5\dot{\eta}_5 + \omega_5^2\eta_5 &= -4.4695\,\ddot{x}_0 \\
\ddot{\eta}_6 + 2\xi_6\omega_6\dot{\eta}_6 + \omega_6^2\eta_6 &= -2.3861\,\ddot{x}_0
\end{aligned}$$

In all six equations $\xi_i = 0.05$. Response of each one of the uncoupled equations is obtained using Newmark's Beta method. The first 10 seconds of response are shown in the following graphs.

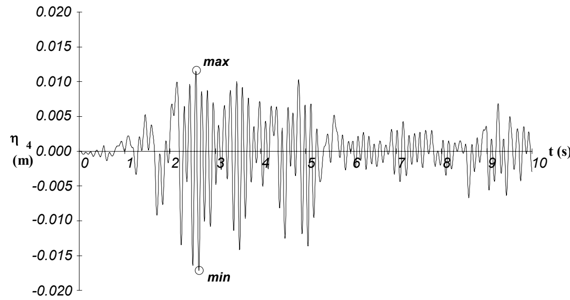


(a) response for η_1 (T_1 = 1.16 s) **(b) response for** η_2 (T_2 = 0.36 s)

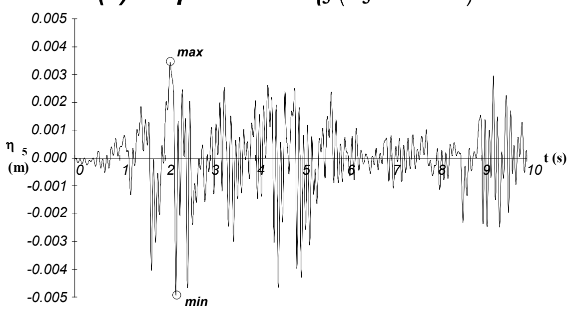
Fig. 11 - Example 5 - Response of the uncoupled coordinates



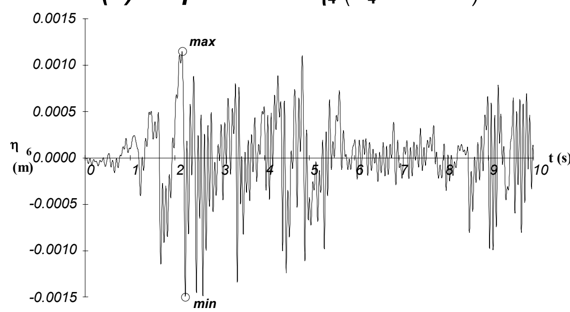
(c) response for η_3 (T_3 = 0.20 s)



(d) response for η_4 (T_4 = 0.13 s)



(e) response for η_5 (T_5 = 0.092 s)



(f) response for η_6 (T_6 = 0.075 s)

Fig. 11 (cont.) - Example 5 - Response of the uncoupled coordinates

The following table contains the response at selected instants, and the extreme values obtained for each uncoupled degree of freedom during the first 10 s of response.

t (s)	η_1 (m)	η_2 (m)	η_3 (m)	η_4 (m)	η_5 (m)	η_6 (m)
2.12	-2.005521	0.120653	0.022884	0.007405	0.003448	0.001047
2.16	-1.454167	0.131596	0.020979	0.009887	0.002831	0.001150
2.22	-0.313056	0.115723	0.009855	-0.005804	-0.002937	-0.001495
2.24	0.044580	0.071039	-0.010748	-0.012843	-0.004919	-0.001187
2.52	2.597408	0.177460	-0.054570	-0.012759	-0.000935	0.000215
2.58	2.305620	0.044364	0.008234	0.010581	-0.001491	0.000439
2.64	1.678734	-0.155214	0.022238	-0.017115	-0.001112	-0.000662
3.04	-3.664644	-0.169206	0.038901	-0.001547	0.000665	0.000280
3.08	-3.545856	-0.153292	-0.001900	0.003579	-0.001228	-0.000357
3.22	-1.871672	0.183567	0.047073	0.005055	0.001766	0.000137
4.58	2.840147	-0.284971	-0.010291	-0.001032	-0.001448	-0.000172
4.76	1.448789	0.295191	-0.005745	-0.002603	0.000228	0.000605
5.90	4.049463	0.020068	0.022552	-0.001531	-0.000257	-0.000036
max	4.049463	0.295191	0.047073	0.010581	0.003448	0.001150
min	-3.664644	-0.284971	-0.054570	-0.017115	-0.004919	-0.001495

Displacements caused by each mode at any instant **t**, is obtained from:

$$\left\{\mathbf{U}^{(i)}\right\} = \left\{\phi^{(i)}\right\}\eta_i(\mathbf{t})$$

Displacements of the structure for the same instant **t**, *are obtained as the sum of the individual contributions of each mode from:*

$$\left\{\mathbf{U}\right\} = \left[\Phi\right]\left\{\eta(\mathbf{t})\right\}$$

For example, for instant **t = 3.08 s**, *displacements in meters contributed by each mode are:*

$$\{U\} = [\Phi]\{\eta\} = \left[\begin{array}{c|c|c|c|c|c} 0.036721 & -0.032775 & 0.029168 & -0.020667 & 0.013049 & -0.005955 \\ \hline 0.033690 & -0.011592 & -0.014245 & 0.032483 & -0.032188 & 0.018512 \\ \hline 0.028524 & 0.014524 & -0.034529 & 0.005317 & 0.028533 & -0.029103 \\ \hline 0.020961 & 0.033322 & -0.005049 & -0.034504 & -0.003317 & 0.033609 \\ \hline 0.012243 & 0.033525 & 0.031633 & 0.006893 & -0.024392 & -0.031454 \\ \hline 0.004460 & 0.015888 & 0.025184 & 0.034025 & 0.035774 & 0.023711 \end{array}\right] \left\{\begin{array}{c} -3.5459 \\ \hline -0.15329 \\ \hline -0.001903 \\ \hline 0.0035796 \\ \hline -0.0012279 \\ \hline -0.00035675 \end{array}\right\}$$

$$= \left\{\begin{array}{c} -0.130210 \\ \hline -0.119460 \\ \hline -0.101140 \\ \hline -0.074326 \\ \hline -0.043414 \\ \hline -0.015815 \end{array}\right\} + \left\{\begin{array}{c} 0.0050242 \\ \hline 0.0017770 \\ \hline -0.0022265 \\ \hline -0.0051079 \\ \hline -0.0051391 \\ \hline -0.0024355 \end{array}\right\} + \left\{\begin{array}{c} -0.000055427 \\ \hline 0.000027070 \\ \hline 0.000065614 \\ \hline 0.000009595 \\ \hline -0.000060111 \\ \hline -0.000047856 \end{array}\right\} + \left\{\begin{array}{c} -0.000073981 \\ \hline 0.000116280 \\ \hline 0.000019033 \\ \hline -0.000123510 \\ \hline 0.000024673 \\ \hline 0.000121800 \end{array}\right\} + \left\{\begin{array}{c} -0.000016023 \\ \hline 0.000039525 \\ \hline -0.000035037 \\ \hline 0.000004073 \\ \hline 0.000029951 \\ \hline -0.000043928 \end{array}\right\} + \left\{\begin{array}{c} 0.000002124 \\ \hline -0.000006604 \\ \hline 0.000010383 \\ \hline -0.000011990 \\ \hline 0.000011221 \\ \hline -0.000008459 \end{array}\right\}$$

$$= \left\{\begin{array}{c} -0.12533 \\ \hline -0.11751 \\ \hline -0.10331 \\ \hline -0.07956 \\ \hline -0.04855 \\ \hline -0.01823 \end{array}\right\}$$

To find the forces imposed by the ground motions at the same instant **t = 3.08 s**, *the stiffness matrix of the structure is multiplied by the displacement just obtained for that instant:*

$$\left\{\mathbf{F}\right\} = \left[\mathbf{K}_\mathbf{E}\right]\left\{\mathbf{U}\right\}$$

This operation can be made for each mode independently, thus obtaining the contribution of the total internal forces caused by each one:

$$\left\{\mathbf{F}^{\mathbf{mod}}\right\} = \left[\mathbf{K}_\mathbf{E}\right]\left[\mathbf{U}^{(1)} \mid \mathbf{U}^{(2)} \mid \cdots \mid \mathbf{U}^{(6)}\right] = \left[\mathbf{F}^{(1)} \mid \mathbf{F}^{(2)} \mid \cdots \mid \mathbf{F}^{(6)}\right]$$

The force contribution in kN for each mode at instant **t = 3.08 s**, *is:*

$$\left\{F^{mod}\right\} = \left[K_E\right]\left\{U^{mod}\right\} = \begin{bmatrix} -970.27 & 388.19 & -13.82 & -47.24 & -19.22 & 3.87 \\ -890.19 & 137.30 & 6.75 & 74.25 & 47.42 & -12.03 \\ -753.69 & -172.02 & 16.36 & 12.15 & -42.04 & 18.91 \\ -553.86 & -394.65 & 2.39 & -78.87 & 4.89 & -21.84 \\ -323.51 & -397.07 & -14.99 & 15.76 & 35.93 & 20.44 \\ -117.85 & -188.18 & -11.93 & 77.77 & -52.70 & -15.41 \end{bmatrix}$$

Total forces in kN for instant **t = 3.08 s**, are:

$$\left\{F\right\} = \left[K_E\right]\left\{U\right\} = \begin{Bmatrix} -658.49 \\ -636.50 \\ -920.33 \\ -1041.90 \\ -663.43 \\ -308.29 \end{Bmatrix}$$

Base shear contributed by each mode, also in kN, at instant **t = 3.08 s**, is obtained from:

$$\left\{V\right\} = \left\{1\right\}^T \left\{F^{mod}\right\}$$

$$= \left\{1 \mid 1 \mid 1 \mid 1 \mid 1 \mid 1\right\} \begin{bmatrix} -970.27 & 388.19 & -13.82 & -47.24 & -19.22 & 3.87 \\ -890.19 & 137.30 & 6.75 & 74.25 & 47.42 & -12.03 \\ -753.69 & -172.02 & 16.36 & 12.15 & -42.04 & 18.91 \\ -553.86 & -394.65 & 2.39 & -78.87 & 4.89 & -21.84 \\ -323.51 & -397.07 & -14.99 & 15.76 & 35.93 & 20.44 \\ -117.85 & -188.18 & -11.93 & 77.77 & -52.70 & -15.41 \end{bmatrix}$$

$$= \left\{-3609.37 \mid -626.43 \mid -15.24 \mid 53.82 \mid -25.72 \mid -6.06\right\}$$

The total base shear in kN at instant **t = 3.08 s**, is obtained as:

$$V = \left\{1\right\}^T \left\{F\right\} = \left\{1 \mid 1 \mid 1 \mid 1 \mid 1 \mid 1\right\} \begin{Bmatrix} -658.49 \\ -636.50 \\ -920.33 \\ -1041.90 \\ -663.43 \\ -308.29 \end{Bmatrix} = -4229.0$$

The overturning moment contributed by each mode in kN · m, for instant **t = 3.08 s**, is obtained from:

$$\{\mathbf{M}\} = \{\mathbf{h}\}^{T}\{\mathbf{F}^{\mathbf{mod}}\}$$

$$= \{18 \mid 15 \mid 12 \mid 9 \mid 6 \mid 3\}\begin{bmatrix} -970.27 & 388.19 & -13.82 & -47.24 & -19.22 & 3.87 \\ -890.19 & 137.30 & 6.75 & 74.25 & 47.42 & -12.03 \\ -753.69 & -172.02 & 16.36 & 12.15 & -42.04 & 18.91 \\ -553.86 & -394.65 & 2.39 & -78.87 & 4.89 & -21.84 \\ -323.51 & -397.07 & -14.99 & 15.76 & 35.93 & 20.44 \\ -117.85 & -188.18 & -11.93 & 77.77 & -52.70 & -15.41 \end{bmatrix}$$

$$= \{-47141 \mid 483.7 \mid -55.4 \mid 27.3 \mid -37.7 \mid -4.0\}$$

The total overturning moment in kN · m at instant **t = 3.08 s**, is obtained from:

$$\mathbf{M} = \{\mathbf{h}\}^{T}\{\mathbf{F}\} = \{18 \mid 15 \mid 12 \mid 9 \mid 6 \mid 3\}\begin{Bmatrix} -658.49 \\ -636.50 \\ -920.33 \\ -1041.90 \\ -663.43 \\ -308.29 \end{Bmatrix} = -46727$$

The same procedures can be used to obtain the response at any instant. If this is performed systematically, results such as shown in Fig. 12 are obtained. There the displacement response for the roof of the building is shown for the first 15 sec. of the to the NS component of El Centro record. From this figure, it is evident that the significant portion of the response is contributed solely by the first two modes, with se second contributing marginally.

Roof displacements

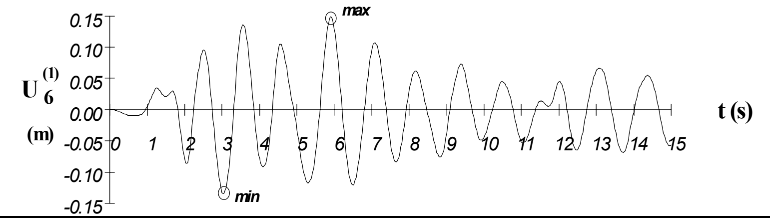
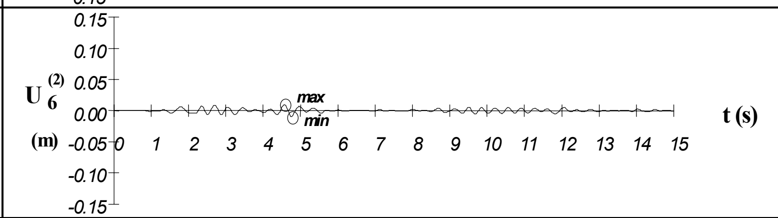
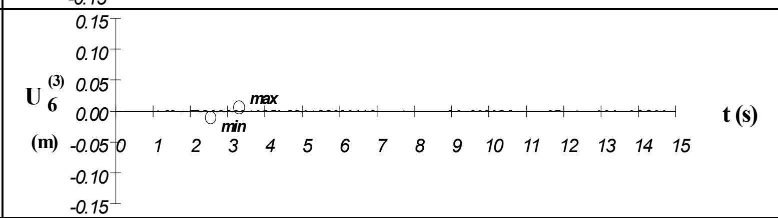
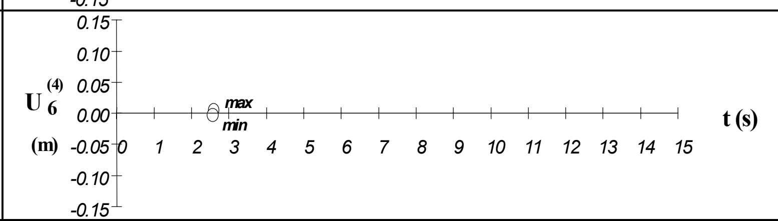
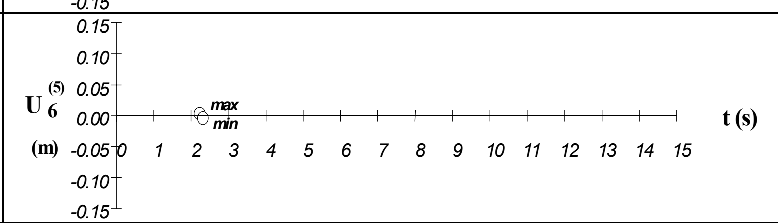
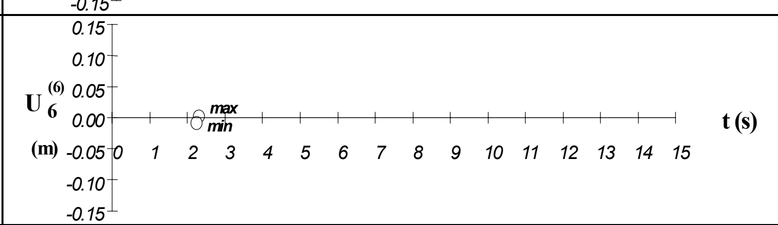
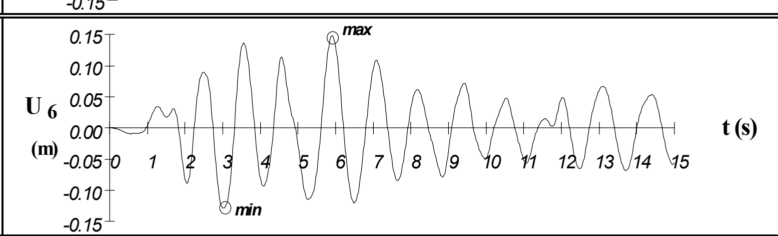
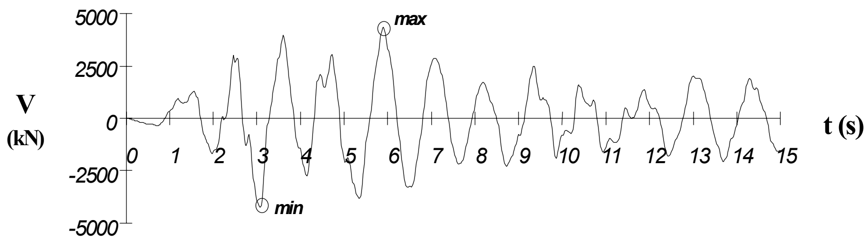
Mode 1		max = 0.14870 m min = -0.13456 m
Mode 2		max = 0.0093399 m min = -0.0096749 m
Mode 3		max = 0.0013730 m min = -0.0015917 m
Mode 4		max = 0.00035371 m min = -0.00021867 m
Mode 5		max = 0.00004500 m min = -0.00006419 m
Mode 6		max = 0.0000089 m min = -0.0000068 m
Total		max = 0.148729 m min = -0.128367 m

Fig. 12 - Example 5 - Roof displacements from each mode and total response

Fig. 13 shows the variation of the base shear of the building during the first 15 sec. of response to the NS component of El Centro record.

Base shear



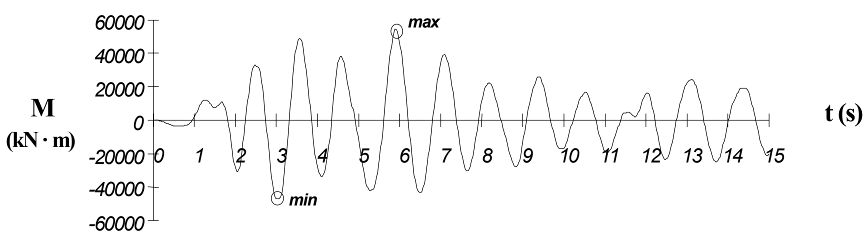
max = 4355.8 kN

min = -4229.1 kN

Fig. 13 - Example 5 - Base shear of the structure

Fig. 14 shows the variation of overturning moment for the first 15 sec. of response to the NS component of El Centro record.

Overturning moment



max = 54406 kN·m

min = -47037 kN·m

Fig. 14 - Example 5 - Overturning moment of the structure

■

Modal spectral analysis

Instead of carrying all the computations involved in obtaining the step-by-step response of the structure as shown in the previous example, an alternative is to use the displacement response spectrum of the ground motion. The displacement response spectrum, $\mathbf{S_d(T,\xi)}$, is the collection of maximum displacements obtained by single degree of freedom systems having period $\mathbf{T}$ and damping coefficient ξ, when subjected to the ground motion record.

Then, the maximum displacement that an uncoupled degree of freedom of the structure can have can be obtained by multiplying the participation coefficient corresponding to the equation by the value read from response spectrum for the vibration period of the equation and the corresponding damping coefficient. This can be stated as:

$$(\eta_i)_{max} = \left|\alpha_i \cdot \mathbf{S_d}\left(\mathbf{T_i}, \xi_i\right)\right| \tag{106}$$

Where $\mathbf{T_i} = \mathbf{2}\pi/\omega_i$ and ξ_i correspond to the value of the modal damping for that mode in particular. If only the acceleration response spectra in known, then the maximum value for the displacement of the uncoupled degree of freedom can be obtained from:

$$(\eta_i)_{max} = \left| \alpha_i \cdot \frac{1}{\omega_i^2} \cdot S_a(T_i, \xi_i) \right| = \left| \alpha_i \cdot \frac{T_i^2}{4\pi^2} \cdot S_a(T_i, \xi_i) \right| \tag{107}$$

The values of maximum displacement of each period of vibration and damping that are collected in the response spectra do not occur at the same time. Therefore, their introduction as a replacement for the values obtained from the step-by-step and their use in computing the displacements and forces induced in the structure has the drawback that we would be adding values of response that didn't occur at the same instant in time. This means that the direct use of the operation implicit in

$$\begin{aligned} \{U\} = [\Phi]\{\eta\} &= \sum_{i=1}^{n} \{\phi^{(i)}\} \eta_i(t) \\ &= \{\phi^{(1)}\}\eta_1(t) + \{\phi^{(2)}\}\eta_2(t) + \cdots + \{\phi^{(n)}\}\eta_n(t) \\ &= \{U^{(1)}\} + \{U^{(2)}\} + \cdots + \{U^{(n)}\} \end{aligned} \tag{108}$$

would not be true, because we would be adding modal displacements that occurred at different instants in time plus not taking into account the proper sign of the response because by definition only absolute values of response are collected in the spectra.

In principle the modal responses obtained using the spectral value are correct since they describe the maximum value that is possible to achieve, simply we are not taking into account the proper sign (negative or positive). The main difficulty rests with the lack of simultaneity. The solution is then, to use one of the *modal combination schemes* that permit the combination of the modes to lead to reasonable results. Lets first study how to obtain the individual modal spectral response parameters without combining them.

The maximum displacements that the structure can have for each individual mode, for example mode ($\mathbf{i}$), may be obtained from:

$$\{U_{mod}^{(i)}\} = \{\phi^{(i)}\} \cdot (\eta_i)_{max} = \{\phi^{(i)}\} \cdot \left| \alpha_i \cdot S_d(T_i, \xi_i) \right| \tag{109}$$

In Eq. (109) it must be take into account that the result multiplied by (-1) is also feasible since it corresponds to a description of an alternating movement. This possibility of positive or negative response is present in all forms of modal spectral response.

For each individual mode (**i**), the maximum lateral forces that can be developed in the structure may be obtained by multiplying the modal spectral displacements by the stiffness matrix of the structure:

$$\left\{\mathbf{F}_{\mathbf{mod}}^{(\mathbf{i})}\right\} = \left[\mathbf{K}\right]\left\{\mathbf{U}_{\mathbf{mod}}^{(\mathbf{i})}\right\} = \left[\mathbf{K}\right]\left\{\phi^{(\mathbf{i})}\right\}\cdot\left(\eta_{\mathbf{i}}\right)_{\mathbf{max}} = \left[\mathbf{K}\right]\left\{\phi^{(\mathbf{i})}\right\}\cdot\left|\alpha_{\mathbf{i}}\cdot\mathbf{S}_{\mathbf{d}}(\mathbf{T}_{\mathbf{i}},\xi_{\mathbf{i}}\right| \qquad (110)$$

These modal spectral forces can be treated as a set of static lateral forces and from a conventional static analysis obtain the internal forces and displacements of the structure caused by mode **(i)**. The internal forces and displacements may be obtained also employing directly the modal spectral displacements from Eq. (109). The two alternatives lead to the same results.

Now we have available, for each individual mode, all parameters of interest such as story drift, overturning moment, and internal forces for the members of the structure. There will be as many sets of these parameters as modes of the structure. At this point, it would be proper to combine them using one of the modal combination schemes.

Example 6

Lets rework Example 5 using the displacement response spectra of the El Centro record.

The results are the same up to the point where the dynamic equilibrium equations were uncoupled.

The uncoupled vibration equations are:

$$\begin{aligned}
\ddot{\eta}_1 + 2\xi_1\omega_1\dot{\eta}_1 + \omega_1^2\eta_1 &= -34.970\,\ddot{x}_0 \\
\ddot{\eta}_2 + 2\xi_2\omega_2\dot{\eta}_2 + \omega_2^2\eta_2 &= -13.540\,\ddot{x}_0 \\
\ddot{\eta}_3 + 2\xi_3\omega_3\dot{\eta}_3 + \omega_3^2\eta_3 &= -8.2331\,\ddot{x}_0 \\
\ddot{\eta}_4 + 2\xi_4\omega_4\dot{\eta}_4 + \omega_4^2\eta_4 &= -6.0279\,\ddot{x}_0 \\
\ddot{\eta}_5 + 2\xi_5\omega_5\dot{\eta}_5 + \omega_5^2\eta_5 &= -4.4695\,\ddot{x}_0 \\
\ddot{\eta}_6 + 2\xi_6\omega_6\dot{\eta}_6 + \omega_6^2\eta_6 &= -2.3861\,\ddot{x}_0
\end{aligned}$$

In all of them, as stated by the problem, $\xi_i = 0.05$.

The response for each of the uncoupled equations is obtained using the displacement response spectra for the N-S component of the El Centro record. En la Fig. 15 shows the spectrum and period fro each mode and the displacement read from the spectrum for each period.

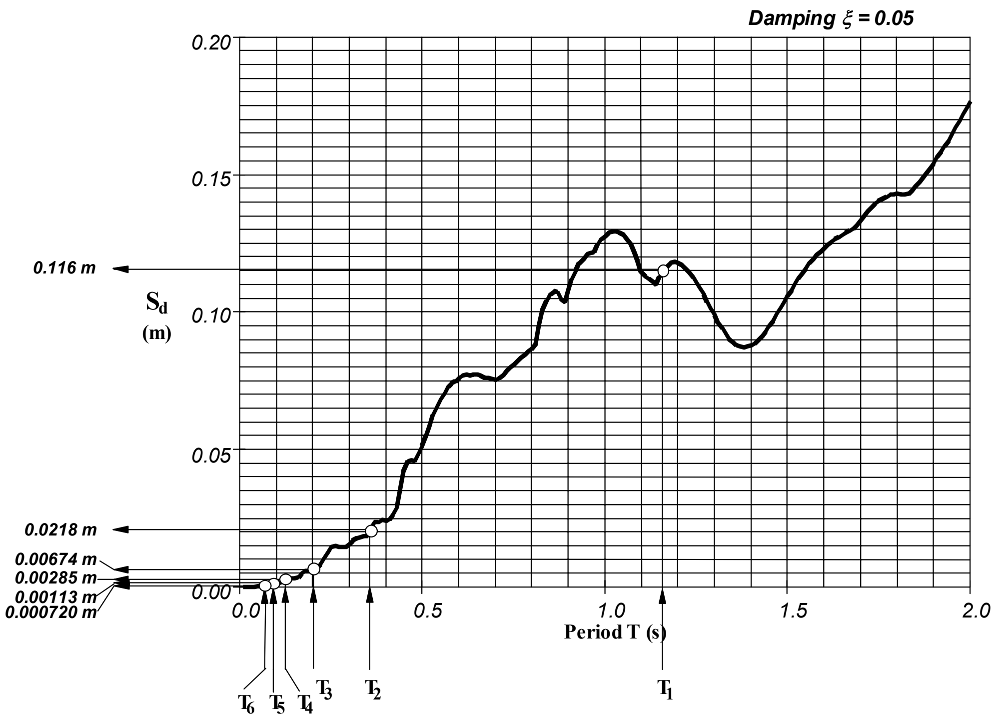


Fig. 15 - Example 6 - Displacement response spectrum for El Centro NS record

Table 1 - Example 6 - Values read from the displacement spectrum

Mode	T_i (s)	$S_d(T_i,\xi_i)$ (m)
1	1.16	0.116
2	0.36	0.0218
3	0.20	0.00674
4	0.13	0.00285
5	0.092	0.00113
6	0.075	0.000720

With this information, it is possible to compute the maximum displacement that the uncoupled degrees of freedom can attain:

Table 2 - Example 6 - Maximum displacement values for the uncoupled degrees of freedom

Mode	α_i	$S_d(T_i,\xi_i)$ (m)	$(\eta_i)_{max} = \alpha_i \times S_d(T_i,\xi_i)$ (m)
1	34.970	0.116	4.0495
2	13.540	0.0218	0.29571
3	8.233	0.00674	0.055458
4	6.028	0.00285	0.017155
5	4.469	0.00113	0.0050639
6	2.386	0.000710	0.0017170

Maximum modal displacements (m)

The maximum displacements for each mode are obtained from:

$$\left\{\mathbf{U}_{\mathbf{mod}}^{(\mathbf{i})}\right\} = \left\{\phi^{(\mathbf{i})}\right\}\left(\eta_{\mathbf{i}}\right)_{\mathbf{max}}$$

These results can be computed for all the modes at the same time by introducing the values of $(\eta_i)_{max}$ in the diagonal of a square matrix $[\mathbf{H_{mod}}]$ y and performing the operation:

$$\left[\mathbf{U}_{\mathbf{mod}}\right] = \left[\Phi\right]\left[\mathrm{H}_{\mathbf{mod}}\right] = \left[\left\{\mathbf{U}_{\mathbf{mod}}^{(1)}\right\} \middle| \left\{\mathbf{U}_{\mathbf{mod}}^{(2)}\right\} \middle| \cdots \middle| \left\{\mathbf{U}_{\mathbf{mod}}^{(\mathbf{n})}\right\}\right]$$

In present case matrix $[\mathbf{H_{mod}}]$ has the following form:

$$\left[\mathrm{H}_{\mathbf{mod}}\right] = \begin{bmatrix} (\eta_1)_{\mathbf{max}} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & (\eta_2)_{\mathbf{max}} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & (\eta_3)_{\mathbf{max}} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & (\eta_4)_{\mathbf{max}} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & (\eta_5)_{\mathbf{max}} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & (\eta_6)_{\mathbf{max}} \end{bmatrix}$$

And replacing the appropriate values from Table 2:

$$\left[\mathrm{H}_{\mathbf{mod}}\right] = \begin{bmatrix} \mathbf{4.0495} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0.29571} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0.0055458} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0.015155} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0.0050639} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0.001717} \end{bmatrix}$$

The values for $[\mathbf{U_{mod}}]$ are:

$$\left[\mathbf{U}_{\mathbf{mod}}\right] = \left[\Phi\right]\left[\mathrm{H}_{\mathbf{mod}}\right] = \begin{array}{cc} \begin{array}{cccccc} \left\{\mathbf{U}_{\mathbf{mod}}^{(1)}\right\} & \left\{\mathbf{U}_{\mathbf{mod}}^{(2)}\right\} & \left\{\mathbf{U}_{\mathbf{mod}}^{(3)}\right\} & \left\{\mathbf{U}_{\mathbf{mod}}^{(4)}\right\} & \left\{\mathbf{U}_{\mathbf{mod}}^{(5)}\right\} & \left\{\mathbf{U}_{\mathbf{mod}}^{(6)}\right\} \end{array} & \downarrow \mathbf{dof} \\ \begin{bmatrix} \mathbf{0.148703} & \mathbf{-0.009692} & \mathbf{0.001618} & \mathbf{-0.000355} & \mathbf{0.000066} & \mathbf{-0.000010} \\ \mathbf{0.136429} & \mathbf{-0.003428} & \mathbf{-0.000790} & \mathbf{0.000557} & \mathbf{-0.000163} & \mathbf{0.000032} \\ \mathbf{0.115519} & \mathbf{0.004295} & \mathbf{-0.001915} & \mathbf{0.000091} & \mathbf{0.000144} & \mathbf{-0.000050} \\ \mathbf{0.084882} & \mathbf{0.009854} & \mathbf{-0.000280} & \mathbf{-0.000592} & \mathbf{-0.000017} & \mathbf{0.000058} \\ \mathbf{0.049588} & \mathbf{0.009914} & \mathbf{0.001754} & \mathbf{0.000118} & \mathbf{-0.000124} & \mathbf{-0.000054} \\ \mathbf{0.018061} & \mathbf{0.004698} & \mathbf{0.001397} & \mathbf{0.000584} & \mathbf{0.000181} & \mathbf{0.000041} \end{bmatrix} & \begin{array}{c} \mathbf{U}_6 \\ \mathbf{U}_5 \\ \mathbf{U}_4 \\ \mathbf{U}_3 \\ \mathbf{U}_2 \\ \mathbf{U}_1 \end{array} \end{array}$$

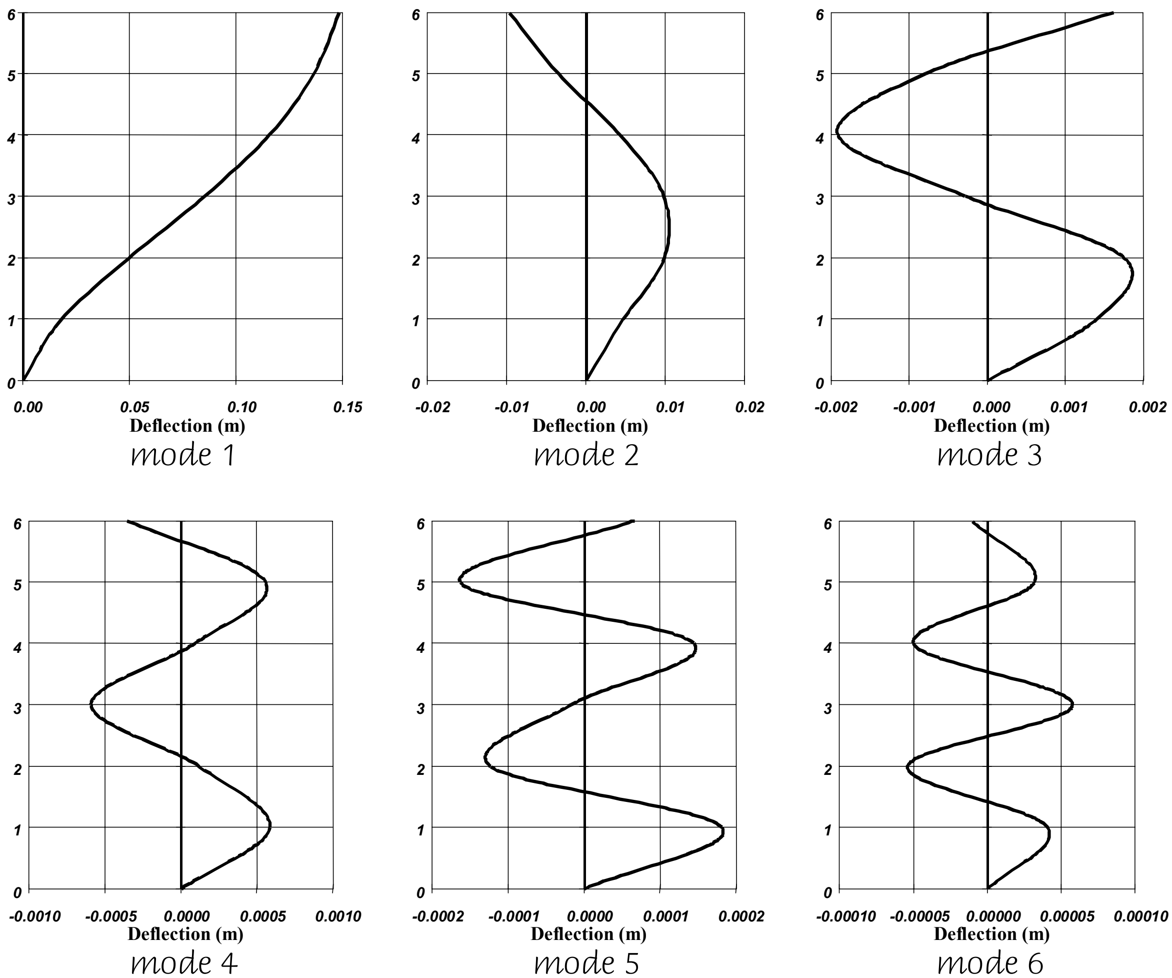


Fig. 16 - Example 6 - Maximum lateral displacements for each mode

Maximum story drift as a percentage of story height (%h)

Using the displacements just computed the story drift for each story and mode could be computed as the algebraic difference of the displacement of two consecutive stories. Drift is usually expressed as percentage of the inter-story height.

Table 3 - Example 6 - Maximum story drift values, as a percentage of story height

story	mode 1	mode 2	mode 3	mode 4	mode 5	mode 6
6	0.409%	-0.209%	0.080%	-0.030%	0.008%	-0.001%
5	0.697%	-0.257%	0.037%	0.016%	-0.010%	0.003%
4	1.021%	-0.185%	-0.054%	0.023%	0.005%	-0.004%
3	1.177%	-0.002%	-0.068%	-0.024%	0.004%	0.004%
2	1.051%	0.174%	0.012%	-0.016%	-0.010%	-0.003%
1	0.602%	0.157%	0.047%	0.019%	0.006%	0.001%

Next figure shows the story drifts for each mode:

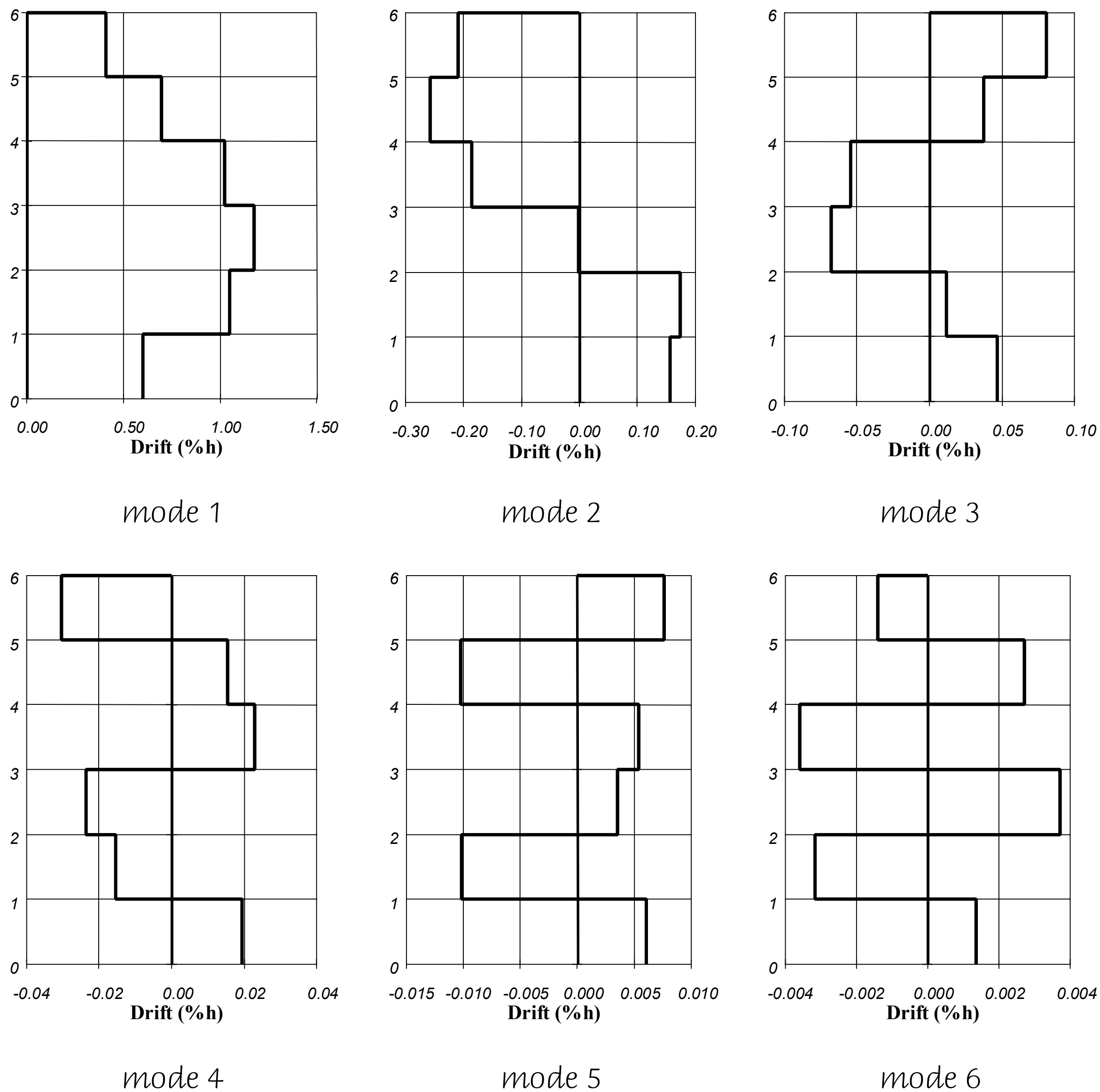


Fig. 17 - Example 6 - Maximum story drift (%h) for each mode

Maximum modal lateral forces (kN)

To obtain the maximum modal lateral forces imposed on the structure by the ground motions the stiffness matrix of the structure is multiplied by the modal lateral displacements. Results are obtained in kN.

$$\left[\mathbf{F}_{\mathbf{mod}}\right]=\left[\mathbf{K}_{\mathbf{E}}\right]\left[\mathbf{U}_{\mathbf{mod}}^{(1)} \mid \mathbf{U}_{\mathbf{mod}}^{(2)} \mid \cdots \mid \mathbf{U}_{\mathbf{mod}}^{(6)}\right]=\left[\mathbf{F}_{\mathbf{mod}}^{(1)} \mid \mathbf{F}_{\mathbf{mod}}^{(2)} \mid \cdots \mid \mathbf{F}_{\mathbf{mod}}^{(6)}\right]$$

$$
\begin{array}{cc}
 & \begin{array}{cccccc} \left\{\mathbf{F}_{\mathbf{mod}}^{(1)}\right\} & \left\{\mathbf{F}_{\mathbf{mod}}^{(2)}\right\} & \left\{\mathbf{F}_{\mathbf{mod}}^{(3)}\right\} & \left\{\mathbf{F}_{\mathbf{mod}}^{(4)}\right\} & \left\{\mathbf{F}_{\mathbf{mod}}^{(5)}\right\} & \left\{\mathbf{F}_{\mathbf{mod}}^{(6)}\right\} \end{array} \quad \downarrow \textbf{dof} \\
\left[\mathbf{F}_{\mathbf{mod}}\right]=\left[\mathbf{K}_{\mathbf{E}}\right]\left[\mathbf{U}_{\mathbf{mod}}\right]= & \left[\begin{array}{r|r|r|r|r|r} \mathbf{1108.3} & \mathbf{-748.9} & \mathbf{403.3} & \mathbf{-226.4} & \mathbf{79.3} & \mathbf{-18.6} \\ \hline \mathbf{1016.2} & \mathbf{-264.8} & \mathbf{-196.9} & \mathbf{355.8} & \mathbf{-195.6} & \mathbf{57.9} \\ \hline \mathbf{860.2} & \mathbf{331.8} & \mathbf{-477.4} & \mathbf{58.2} & \mathbf{173.4} & \mathbf{-91.0} \\ \hline \mathbf{632.9} & \mathbf{761.5} & \mathbf{-69.8} & \mathbf{-378.0} & \mathbf{-20.2} & \mathbf{105.1} \\ \hline \mathbf{369.4} & \mathbf{765.9} & \mathbf{437.3} & \mathbf{75.5} & \mathbf{-148.2} & \mathbf{-98.4} \\ \hline \mathbf{135.1} & \mathbf{363.0} & \mathbf{348.2} & \mathbf{372.7} & \mathbf{217.3} & \mathbf{74.1} \end{array}\right] \begin{array}{c} \mathbf{F_6} \\ \mathbf{F_5} \\ \mathbf{F_4} \\ \mathbf{F_3} \\ \mathbf{F_2} \\ \mathbf{F_1} \end{array}
\end{array}
$$

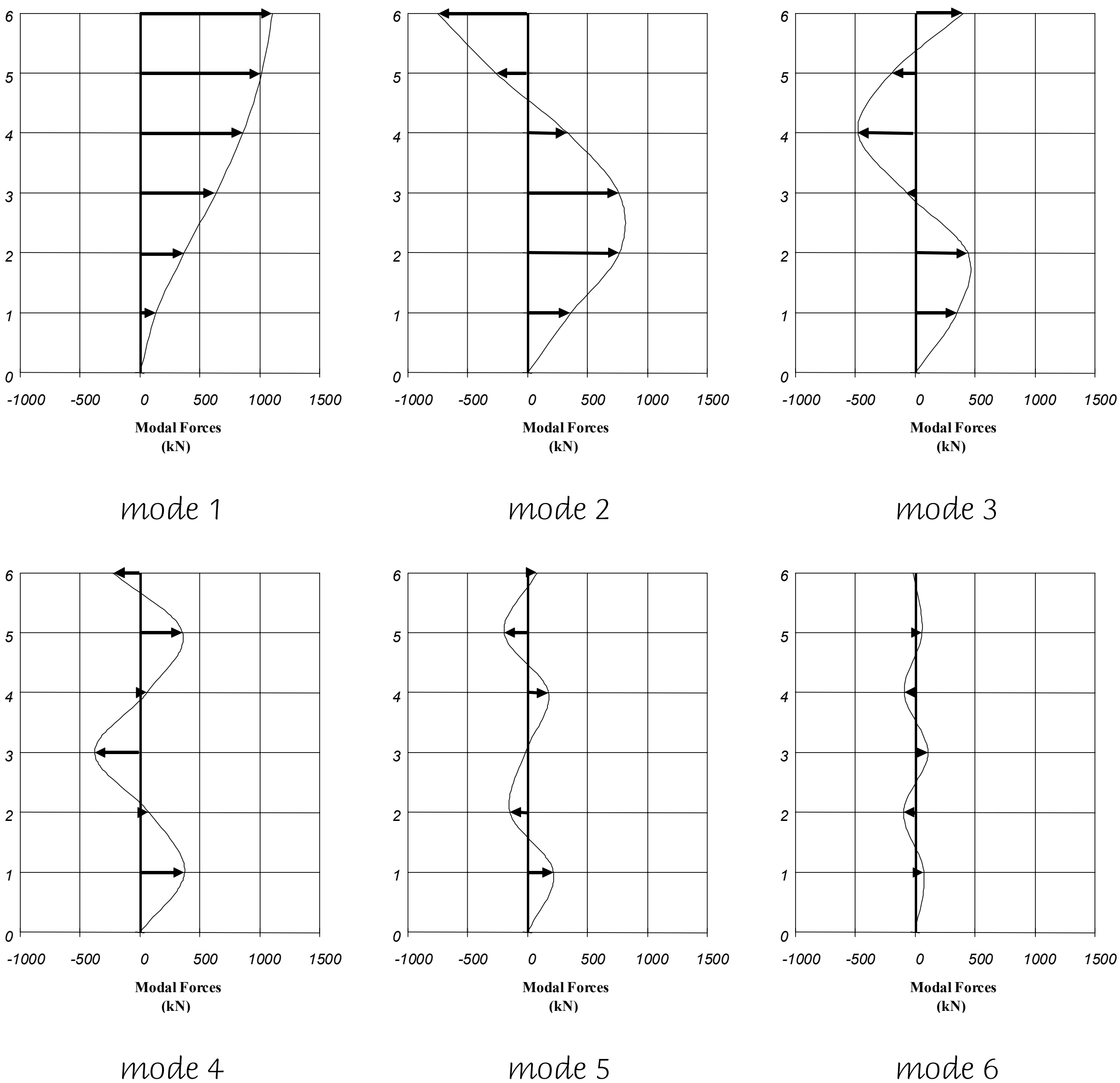


Fig 18 - Example 6 - Maximum modal forces for each mode

<u>Maximum modal story shear (kN)</u>

The maximum modal story shear is obtained from $\mathbf{V}_{\mathbf{j}}^{(\mathbf{i})}=\sum_{\mathbf{k}=\mathbf{j}}^{\mathbf{n}} \mathbf{F}_{\mathbf{k}}^{(\mathbf{i})}$

Table 4 - Example 6 - Maximum modal values for story shear

story	$V^{(1)}_{mod}$ (kN)	$V^{(2)}_{mod}$ (kN)	$V^{(3)}_{mod}$ (kN)	$V^{(4)}_{mod}$ (kN)	$V^{(5)}_{mod}$ (kN)	$V^{(6)}_{mod}$ (kN)
6	1108.3	-748.9	403.3	-226.4	79.3	-18.6
5	2124.6	-1013.7	206.3	129.4	-116.3	39.3
4	2984.8	-681.9	-271.0	187.7	57.1	-51.7
3	3617.6	79.6	-340.9	-190.3	36.9	53.4
2	3987.0	845.5	96.5	-114.8	-111.3	-45.0
1	4122.1	1208.5	444.6	257.9	106.1	29.1
0	4122.1	1208.5	444.6	257.9	106.1	29.1

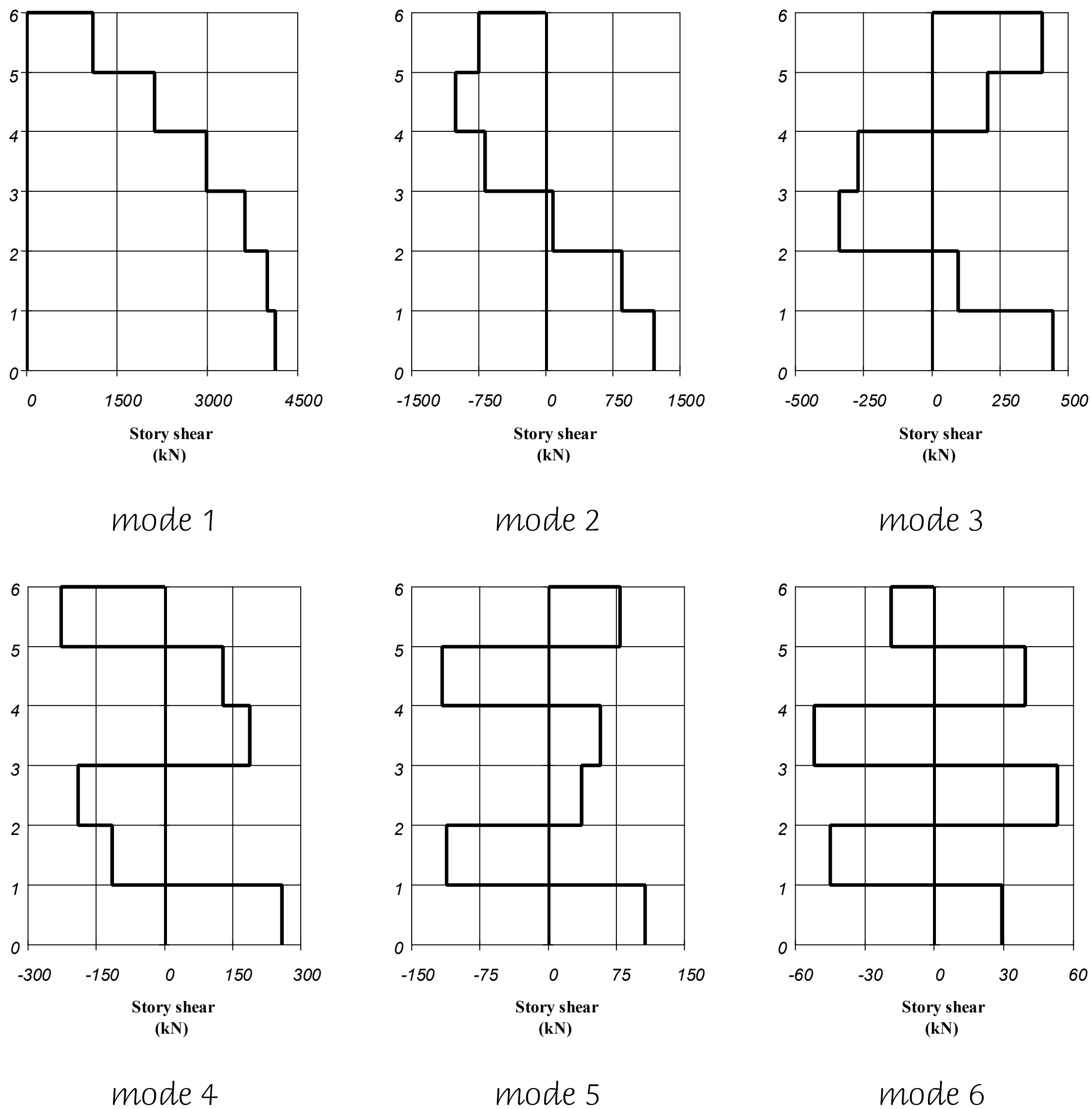


Fig. 19 - Example 6 - Maximum story shear for each mode

Base shear (kN)

The base shear in kN for each mode is obtained from

$$\{\mathbf{V_{mod}}\} = \{1\}^T [\mathbf{F_{mod}}] = \{1 \mid 1 \mid 1 \mid 1 \mid 1 \mid 1\} \begin{bmatrix} 1108.3 & -748.9 & 403.3 & -226.4 & 79.3 & -18.6 \\ 1016.2 & -264.8 & -196.9 & 355.8 & -195.6 & 57.9 \\ 860.2 & 331.8 & -477.4 & 58.2 & 173.4 & -91.0 \\ 632.9 & 761.5 & -69.8 & -378.0 & -20.2 & 105.1 \\ 369.4 & 765.9 & 437.3 & 75.5 & -148.2 & -98.4 \\ 135.1 & 363.0 & 348.2 & 372.7 & 217.3 & 74.1 \end{bmatrix}$$

$$= \{\underset{\mathbf{V_{mod}^{(1)}}}{4122.1} \mid \underset{\mathbf{V_{mod}^{(2)}}}{1208.5} \mid \underset{\mathbf{V_{mod}^{(3)}}}{444.6} \mid \underset{\mathbf{V_{mod}^{(4)}}}{257.9} \mid \underset{\mathbf{V_{mod}^{(5)}}}{106.1} \mid \underset{\mathbf{V_{mod}^{(6)}}}{29.1}\}$$

It is the same value obtained for the first story when the story shears were computed.

Overturning moment (kN · m)

The overturning moment for each story is obtained from $\mathbf{M_j^{(i)}} = \sum_{k=j+1}^{n} (\mathbf{h_k} - \mathbf{h_j}) \cdot \mathbf{F_j^{(i)}}$

Table 5 - Example 6 - Maximum story modal overturning moment

story	$\mathbf{M_{mod}^{(1)}}$ (kN · m)	$\mathbf{M_{mod}^{(2)}}$ (kN · m)	$\mathbf{M_{mod}^{(3)}}$ (kN · m)	$\mathbf{M_{mod}^{(4)}}$ (kN · m)	$\mathbf{M_{mod}^{(5)}}$ (kN · m)	$\mathbf{M_{mod}^{(6)}}$ (kN · m)
6	0.0	0.0	0.0	0.0	0.0	0.0
5	3324.9	-2246.7	1209.8	-679.2	237.8	-55.9
4	9698.6	-5287.8	1828.7	-290.9	-111.0	61.9
3	18652.9	-7333.6	1015.6	272.2	60.2	-93.3
2	29505.8	-7094.7	-6.9	-298.7	170.9	66.8
1	41466.8	-4558.2	282.4	-643.1	-162.9	-68.2
0	53833.1	-932.7	1616.3	130.7	155.3	19.2

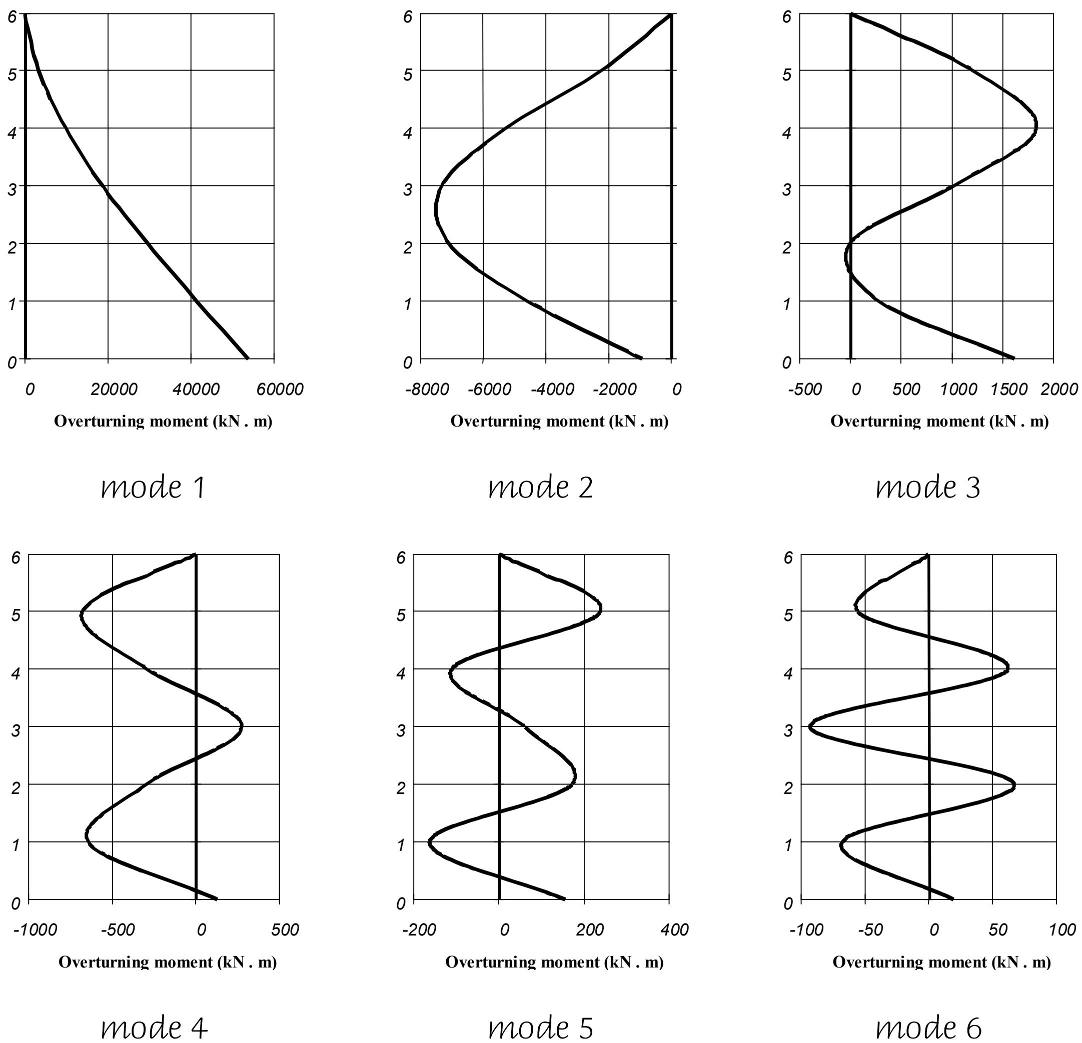


Fig. 20 - Example 6 - Overturning moment for each mode

The maximum overturning moment at the base, in kN · m, contributed by each mode can be obtained from:

$$\{\mathbf{M_{mod}}\} = \{\mathbf{h}\}^T [\mathbf{F_{mod}}] = \{18 \mid 15 \mid 12 \mid 9 \mid 6 \mid 3\} \begin{bmatrix} 1108.3 & -748.9 & 403.3 & -226.4 & 79.3 & -18.6 \\ 1016.2 & -264.8 & -196.9 & 355.8 & -195.6 & 57.9 \\ 860.2 & 331.8 & -477.4 & 58.2 & 173.4 & -91.0 \\ 632.9 & 761.5 & -69.8 & -378.0 & -20.2 & 105.1 \\ 369.4 & 765.9 & 437.3 & 75.5 & -148.2 & -98.4 \\ 135.1 & 363.0 & 348.2 & 372.7 & 217.3 & 74.1 \end{bmatrix}$$

$$= \{\underset{\mathbf{M_{mod}^{(1)}}}{53833} \mid \underset{\mathbf{M_{mod}^{(2)}}}{-933} \mid \underset{\mathbf{M_{mod}^{(3)}}}{1616} \mid \underset{\mathbf{M_{mod}^{(4)}}}{131} \mid \underset{\mathbf{M_{mod}^{(5)}}}{155} \mid \underset{\mathbf{M_{mod}^{(6)}}}{19}\}$$

This is the same result obtained for the overturning moment previously.

In Example 5 the step-by-step response of the building was obtained for the same earthquake record used to compute the spectrum in this example, it is interesting to make some comparisons of the results obtained in both cases. Table 6 lists the values obtained in Example 5 and Example 6 for each of the uncoupled degrees of freedom.

Table 6 - Example 6 - Comparison of values obtained in Examples 5 and 6

Uncoupled degree of freedom	Example 5			Example 6
		η_i (m)	t (s)	$(\eta_i)_{max} = \alpha_i \times S_d(T_i, \xi_i)$ (m)
η_1	max	4.049463	5.90	4.0495
	min	-3.664644	3.04	
η_2	max	0.295191	4.76	0.29571
	min	-0.284971	4.58	
η_3	max	0.047073	3.22	0.055458
	min	-0.054570	2.52	
η_4	max	0.010581	2.58	0.017155
	min	-0.017115	2.64	
η_5	max	0.003448	2.12	0.0050639
	min	-0.004919	2.24	
η_6	max	0.001150	2.16	0.0017170
	min	-0.001495	2.22	

As can be seen, the results are essentially the same, and the differences obey to precision rounding in the numerical procedures because the algorithm employed to obtain the response is different from the one used to compute the spectrum. It should be noted that the maximum values for each uncoupled degree of freedom in Example 5 were obtained at different time instants It should also be noted that the maximum value obtained from the spectrum in some cases correspond to the maximum value and in some to the minimum obtained in the step-by-step procedure, this is because the value carried by the spectrum is the absolute value.

The maximum lateral displacement of the roof obtained in Example 5 was 0.149 m. The algebraic sum of the values obtained for the 6th story in Example 6 is 0.140 m, and the sum of the absolute values is 0.160 m. The algebraic sum of the modal response underestimates the value obtained using a time step-by-step procedure and the sum of the absolute modal values overestimate it.

The maximum value for the base shear of the building obtained in Example 5 using a time step-by-step procedure was 4360 kN. The sum of the maximum modal base shears obtained in Example 6 was 6170 kN. This value overestimates the time step value by a factor of 1.4. In the time step procedure of Example 5 the *base shear is controlled by the first mode with the other modes contributing very little when the first mode peak occurs. For the overturning moment at the base in Example 5 a value of 54,400 kN · m was obtained In Example 6 the algebraic sum of the maximum modal values is 54,800 kN · m, and the sum of the absolute values is 56,690 kN · m. For the overturning moment, the contribution of the higher modes is small in both examples.*

■

Modal combination using the square root of the squares (SRSS)

The most widely known method of modal spectral combination is called *SRSS* or *square root of the sum of the squares*. The method indicates that for any parameter of modal spectral response, $\mathbf{r}$, the maximum credible value of the parameter, $\overline{\mathbf{r}}$, when taking into account $\mathbf{m}$ modal components $\mathbf{r_i}$ is obtained from:

$$\overline{\mathbf{r}} \approx \sqrt{\sum_{\mathbf{i=1}}^{\mathbf{m}} \mathbf{r_i^2}} \tag{111}$$

The SRSS method must be employed using the modal spectral values of the response parameter, $\mathbf{r_i}$. It must be taken into account that the result of the SRSS methods will always give positive values, but in reality, it can be either positive or negative since it is the representation of an oscillatory movement. This is the reason that when earthquake related effects are combined with those from other sources, such as gravity loads, the positive and negative cases of the earthquake forces must be used in the load combination employed in design.

The application procedure for the SRSS method for different parameters of the response will be described for a structure having $\mathbf{p}$ stories and $\mathbf{m}$ modes:

(a) **Maximum credible lateral displacements of the structure** - The maximum modal spectral lateral displacements for each mode $\mathbf{i}$ are obtained from Eq. (109). Then using the SRSS method the maximum credible value of the lateral displacement for story $\mathbf{j}$ would be:

$$\mathbf{U_j^{srss}} = \sqrt{\sum_{\mathbf{i=1}}^{\mathbf{m}} \left(\mathbf{U_j^{(i)}}\right)^2} = \sqrt{\left(\mathbf{U_j^{(1)}}\right)^2 + \left(\mathbf{U_j^{(2)}}\right)^2 + \cdots + \left(\mathbf{U_j^{(m)}}\right)^2} \tag{112}$$

(b) **Maximum credible story drift** - Using the maximum modal spectral displacement, for example for story $\mathbf{j}$ and mode $\mathbf{i}$, $\mathbf{U_j^{(i)}}$, story drift is computes first as the difference of lateral displacements:

$$\Delta_{\mathbf{j}}^{\mathbf{(i)}} = \mathbf{U_{j+1}^{(i)}} - \mathbf{U_j^{(i)}} \tag{113}$$

Then using the SRSS method the maximum credible value of the story drift for story $\mathbf{j}$ would be:

$$\Delta_j^{srss} = \sqrt{\sum_{i=1}^{m}\left(\Delta_j^{(i)}\right)^2} = \sqrt{\left(\Delta_j^{(1)}\right)^2 + \left(\Delta_j^{(2)}\right)^2 + \cdots + \left(\Delta_j^{(m)}\right)^2} \qquad (114)$$

It is important to note that *it would be wrong to compute the maximum credible story drift using values of the displacements that have already been combined*. Thus, under the SRSS methodology, it wouldn't be licit to employ the maximum credible lateral displacements obtained from Eq. (112) to compute the story drift. The reason is that by squaring the displacements in Eq. (112) the sign of the negative displacements would be lost and Eq. (113) would report a lower value for the story drift when there is a sign difference between the two story displacements.

(c) **Maximum credible story shear** - Using Eq. (110) the maximum modal spectral forces for story $\mathbf{k}$ and mode $\mathbf{i}$ are obtained: $\mathbf{F}_k^{(i)}$. Then the modal spectral story shear is obtained from for each story and each mode $\mathbf{i}$:

$$\mathbf{V}_j^{(i)} = \sum_{k=j}^{p} \mathbf{F}_k^{(i)} \qquad (115)$$

Then, using the SRSS procedure the maximum credible story shear is obtained for story $\mathbf{j}$ from:

$$\mathbf{V}_j^{srss} = \sqrt{\sum_{i=1}^{m}\left(\mathbf{V}_j^{(i)}\right)^2} = \sqrt{\left(\mathbf{V}_j^{(1)}\right)^2 + \left(\mathbf{V}_j^{(2)}\right)^2 + \cdots + \left(\mathbf{V}_j^{(m)}\right)^2} \qquad (116)$$

In this case, as for story drift, *it would be wrong to compute the maximum credible story shear using values of the lateral forces that have already been combined*.

(d) **Maximum credible base shear** - First, the modal spectral base shear for each mode $\mathbf{i}$ is obtained from the algebraic sum of the modal spectral story forces, $\mathbf{F}_k^{(i)}$:

$$\mathbf{V}_{mod}^{(i)} = \sum_{k=1}^{p} \mathbf{F}_k^{(i)} \qquad (117)$$

Then, using the SRSS procedure the maximum credible base shear is obtained from:

$$\mathbf{V}^{srss} = \sqrt{\sum_{i=1}^{m}\left(\mathbf{V}_{mod}^{(i)}\right)^2} = \sqrt{\left(\mathbf{V}_{mod}^{(1)}\right)^2 + \left(\mathbf{V}_{mod}^{(2)}\right)^2 + \cdots + \left(\mathbf{V}_{mod}^{(m)}\right)^2} \quad (118)$$

In this case, as for story drift and story shear, *it would be wrong to compute the maximum credible base shear using values of the lateral forces that have already been combined*.

(e) **Maximum credible story overturning moment** - Using Eq. (110) the maximum modal spectral forces for story $\mathbf{k}$ and mode $\mathbf{i}$ are obtained: $\mathbf{F}_k^{(i)}$. Then the modal spectral story overturning moment is obtained from for each story and for mode $\mathbf{i}$ as:

$$\mathbf{M}_j^{(i)} = \sum_{k=j+1}^{p}\left[\left(\mathbf{h}_k - \mathbf{h}_j\right)\cdot \mathbf{F}_k^{(i)}\right] \quad (119)$$

Where $\mathbf{h}_k$ and $\mathbf{h}_j$ are the height measured from the base of stories $\mathbf{k}$ and $\mathbf{j}$, respectively. Using the SRSS method, the maximum credible story overturning moment is obtained from:

$$\mathbf{M}_j^{srss} = \sqrt{\sum_{i=1}^{m}\left(\mathbf{M}_j^{(i)}\right)^2} = \sqrt{\left(\mathbf{M}_j^{(1)}\right)^2 + \left(\mathbf{M}_j^{(2)}\right)^2 + \cdots + \left(\mathbf{M}_j^{(m)}\right)^2} \quad (120)$$

The same warning is warranted: *it would be wrong to compute the maximum credible story overturning moment using values of the lateral forces that have already been combined*.

(f) **Maximum credible base overturning moment** - Using the modal spectral forces for mode $\mathbf{i}$, $\mathbf{F}_k^{(i)}$, the base overturning moment for mode $\mathbf{i}$ is obtained from:

$$\mathbf{M}_{mod}^{(i)} = \sum_{k=1}^{p}\left(\mathbf{h}_k \cdot \mathbf{F}_k^{(i)}\right) \quad (121)$$

The maximum credible base overturning moment is obtained from:

$$\mathbf{M}^{srss} = \sqrt{\sum_{i=1}^{m}\left(\mathbf{M}_{mod}^{(i)}\right)^2} = \sqrt{\left(\mathbf{M}_{mod}^{(1)}\right)^2 + \left(\mathbf{M}_{mod}^{(2)}\right)^2 + \cdots + \left(\mathbf{M}_{mod}^{(m)}\right)^2} \quad (122)$$

The same warning is warranted: *it would be wrong to compute the maximum credible base overturning moment using values of the lateral forces that have already been combined*.

(g) **Static equivalent lateral forces corresponding to the maximum credible modal lateral forces** - For obtaining the internal forces of elements of the structure using a conventional static analysis, it is sometimes convenient to have available a set of static forces that represent the maximum credible spectral effects on the structure. The static equivalent lateral forces are obtained from the maximum credible story forces obtained in (c). The equivalent force at each story would be the difference in story shear between two contiguous stories. At the roof, it is equal to he story shear there. Then, for any story **j** the equivalent static force would be:

$$\mathbf{F_j^E} = \begin{cases} \mathbf{V_j^{max}} & \mathbf{for} \quad \mathbf{j = p} \\ \mathbf{V_j^{max} - V_{j+1}^{max}} & \mathbf{for} \quad \mathbf{j \neq p} \end{cases} \tag{123}$$

Previous presentation of the SRSS method was made having a planar system in mind. For three-dimensional cases some additional combining parameters must be taken into account, related mainly with the employment of different components of the earthquake along the principal axes of the structure. This is beyond the scope of this introductory presentation of multi-degree of freedom systems.

Example 7

Apply the square root of the sum of the squares SRSS procedure to the results obtained in Example 6. The use of the SRSS technique produces the following results:

<u>Maximum credible lateral displacements (m)</u>

The maximum modal displacements were obtained from:

$$\left\{\mathbf{U_{mod}^{(i)}}\right\} = \left\{\phi^{(i)}\right\}\left(\eta_i\right)_{\mathbf{max}}$$

as:

$$\left[\mathbf{U_{mod}}\right] = \left[\Phi\right]\left[\mathrm{H_{mod}}\right] = \begin{array}{c} \begin{array}{cccccc} \{\mathbf{U_{mod}^{(1)}}\} & \{\mathbf{U_{mod}^{(2)}}\} & \{\mathbf{U_{mod}^{(3)}}\} & \{\mathbf{U_{mod}^{(4)}}\} & \{\mathbf{U_{mod}^{(5)}}\} & \{\mathbf{U_{mod}^{(6)}}\} \end{array} \\ \left[\begin{array}{cccccc} 0.148703 & -0.009692 & 0.001618 & -0.000355 & 0.000066 & -0.000010 \\ 0.136429 & -0.003428 & -0.000790 & 0.000557 & -0.000163 & 0.000032 \\ 0.115519 & 0.004295 & -0.001915 & 0.000091 & 0.000144 & -0.000050 \\ 0.084882 & 0.009854 & -0.000280 & -0.000592 & -0.000017 & 0.000058 \\ 0.049588 & 0.009914 & 0.001754 & 0.000118 & -0.000124 & -0.000054 \\ 0.018061 & 0.004698 & 0.001397 & 0.000584 & 0.000181 & 0.000041 \end{array}\right] \end{array} \begin{array}{c} \downarrow \mathbf{dof} \\ \mathbf{U_6} \\ \mathbf{U_5} \\ \mathbf{U_4} \\ \mathbf{U_3} \\ \mathbf{U_2} \\ \mathbf{U_1} \end{array}$$

We now apply the SRSS procedure to each of the row of previous matrix. For example for the roof (6th story):

$$U_6^{max} = \sqrt{(0.148703)^2 + (-0.009692)^2 + (0.001618)^2 + (-0.000355)^2 + (0.000066)^2 + (-0.000010)^2}$$
$$= 0.14903 \text{ m}$$

This value compares fairly well with the value of 0.14873 m obtained from the step-by-step procedure in Example 6. The result, in m, for all stories is:

$$\{U^{srss}\} = \pm \begin{Bmatrix} 0.14903 \\ 0.13648 \\ 0.11560 \\ 0.08545 \\ 0.05059 \\ 0.01872 \end{Bmatrix} \begin{array}{c} \downarrow \text{dof} \\ U_6 \\ U_5 \\ U_4 \\ U_3 \\ U_2 \\ U_1 \end{array}$$

A symbol ± has been introduced to remind that the values obtained from the SRSS procedure may be either positive or negative.

Maximum credible story drift

The modal spectral story drifts are computed from the values shown in **[U_{mod}]**. Using Eq. (113) the following result are obtained:

$$[\Delta_{mod}] = \begin{array}{cccccc|c} \{\Delta_{mod}^{(1)}\} & \{\Delta_{mod}^{(2)}\} & \{\Delta_{mod}^{(3)}\} & \{\Delta_{mod}^{(4)}\} & \{\Delta_{mod}^{(5)}\} & \{\Delta_{mod}^{(6)}\} & \downarrow \text{story} \end{array}$$

$\{\Delta_{mod}^{(1)}\}$	$\{\Delta_{mod}^{(2)}\}$	$\{\Delta_{mod}^{(3)}\}$	$\{\Delta_{mod}^{(4)}\}$	$\{\Delta_{mod}^{(5)}\}$	$\{\Delta_{mod}^{(6)}\}$	↓ story
0.012274	−0.006264	0.002408	−0.000912	0.000229	−0.000042	6
0.020920	−0.007723	0.001125	0.000466	−0.000307	0.000082	5
0.030627	−0.005559	−0.001635	0.000683	0.000161	−0.000108	4
0.035304	−0.000060	−0.002034	−0.000710	0.000107	0.000112	3
0.031517	0.005216	0.000358	−0.000465	−0.000305	−0.000095	2
0.018061	0.004698	0.001397	0.000584	0.000181	0.000041	1

As an example, we now apply the SRSS procedure to the third story:

$$\Delta_3^{srss} = \sqrt{(0.035304)^2 + (-0.000060)^2 + (-0.002034)^2 + (-0.000710)^2 + (0.000107)^2 + (0.000112)^2}$$
$$= 0.03537 \text{ m}$$

And for all stories:

$$\left\{\Delta^{srss}\right\} = \begin{Bmatrix} 0.0140 \\ 0.0223 \\ 0.0312 \\ 0.0354 \\ 0.0320 \\ 0.0188 \end{Bmatrix} m = \begin{array}{c} \\ \left\{\begin{array}{c} 0.47\%h \\ 0.74\%h \\ 1.04\%h \\ 1.18\%h \\ 1.07\%h \\ 0.62\%h \end{array}\right\} \end{array} \begin{array}{c} \downarrow \text{story} \\ 6 \\ 5 \\ 4 \\ 3 \\ 2 \\ 1 \end{array}$$

Now, for the sake of discussion, lets compute erroneously the story drift from lateral displacements already combined, $\{U^{srss}\}$. The following are the results for story drift as a percentage of the story height (%h) thus computed:

$$\left\{\Delta\right\} = \left\{\begin{array}{c} 0.42\%h \\ 0.70\%h \\ 1.00\%h \\ 1.16\%h \\ 1.06\%h \\ 0.62\%h \end{array}\right\} \begin{array}{ll} \downarrow \text{story} & \\ 6 & \Leftarrow \text{wrong result} \\ 5 & \Leftarrow \text{wrong result} \\ 4 & \Leftarrow \text{wrong result} \\ 3 & \Leftarrow \text{wrong result} \\ 2 & \\ 1 & \end{array}$$

Maximum credible story forces (kN)

The maximum modal spectral forces were obtained for each mode in Example 6 multiplying the stiffness matrix by the modal spectral displacements of each mode, obtaining there the following forces in kN:

$$\left[F_{mod}\right] = \left[K_E\right]\left[U^{(1)}_{mod} \mid U^{(2)}_{mod} \mid \cdots \mid U^{(6)}_{mod}\right] = \left[F^{(1)}_{mod} \mid F^{(2)}_{mod} \mid \cdots \mid F^{(6)}_{mod}\right]$$

$$\left[F_{mod}\right] = \left[K_E\right]\left[U_{mod}\right] = \begin{array}{ccccccc} \{F^{(1)}_{mod}\} & \{F^{(2)}_{mod}\} & \{F^{(3)}_{mod}\} & \{F^{(4)}_{mod}\} & \{F^{(5)}_{mod}\} & \{F^{(6)}_{mod}\} & \downarrow \text{dof} \\ \end{array}$$

$\{F^{(1)}_{mod}\}$	$\{F^{(2)}_{mod}\}$	$\{F^{(3)}_{mod}\}$	$\{F^{(4)}_{mod}\}$	$\{F^{(5)}_{mod}\}$	$\{F^{(6)}_{mod}\}$	↓ dof
1108.3	−748.9	403.3	−226.4	79.3	−18.6	F_6
1016.2	−264.8	−196.9	355.8	−195.6	57.9	F_5
860.2	331.8	−477.4	58.2	173.4	−91.0	F_4
632.9	761.5	−69.8	−378.0	−20.2	105.1	F_3
369.4	765.9	437.3	75.5	−148.2	−98.4	F_2
135.1	363.0	348.2	372.7	217.3	74.1	F_1

A sensible recommendation is to keep these modal forces separated by mode and never combine them using SRSS. This way the danger of using the combined forces in the computation of story shears and overturning moments is avoided.

Maximum credible story shear (kN)

The maximum credible modal spectral story shear may be obtained from Eq. (115)

$$\mathbf{V_j^{(i)}} = \sum_{k=j}^{p} \mathbf{F_k^{(i)}}$$

Table 7 - Example 7 - Story shear modal spectral values

story	$\mathbf{V_{mod}^{(1)}}$ (kN)	$\mathbf{V_{mod}^{(2)}}$ (kN)	$\mathbf{V_{mod}^{(3)}}$ (kN)	$\mathbf{V_{mod}^{(4)}}$ (kN)	$\mathbf{V_{mod}^{(5)}}$ (kN)	$\mathbf{V_{mod}^{(6)}}$ (kN)
6	1108.3	-748.9	403.3	-226.4	79.3	-18.6
5	2124.6	-1013.7	206.3	129.4	-116.3	39.3
4	2984.8	-681.9	-271.0	187.7	57.1	-51.7
3	3617.6	79.6	-340.9	-190.3	36.9	53.4
2	3987.0	845.5	96.5	-114.8	-111.3	-45.0
1	4122.1	1208.5	444.6	257.9	106.1	29.1

Applying, for example, the SRSS procedure to the second story, we obtain:

$$\begin{aligned}\mathbf{V_2^{srss}} &= \sqrt{(\mathbf{3987.0})^2 + (\mathbf{845.5})^2 + (\mathbf{96.5})^2 + (\mathbf{-114.8})^2 + (\mathbf{-111.3})^2 + (\mathbf{-45.0})^2} \\ &= \mathbf{4080.2 \quad kN}\end{aligned}$$

The result, in kN, for all stories is

$$\left\{\mathbf{V^{srss}}\right\} = \pm \begin{array}{c} \\ \left\{\begin{array}{c} \mathbf{1417.6} \\ \hline \mathbf{2369.8} \\ \hline \mathbf{3080.3} \\ \hline \mathbf{3640.1} \\ \hline \mathbf{4080.2} \\ \hline \mathbf{4327.6} \end{array}\right\} \end{array} \begin{array}{l} \downarrow \textbf{story} \\ \mathbf{6} \\ \mathbf{5} \\ \mathbf{4} \\ \mathbf{3} \\ \mathbf{2} \\ \mathbf{1} \end{array}$$

Maximum credible base shear

The base shear, in kN, was obtained in Example 6 for each mode as:

$$\{\mathbf{V_{mod}}\} = \{1\}^T [\mathbf{F_{mod}}] = \{1 \mid 1 \mid 1 \mid 1 \mid 1 \mid 1\} \left[\begin{array}{c|c|c|c|c|c} 1108.3 & -748.9 & 403.3 & -226.4 & 79.3 & -18.6 \\ \hline 1016.2 & -264.8 & -196.9 & 355.8 & -195.6 & 57.9 \\ \hline 860.2 & 331.8 & -477.4 & 58.2 & 173.4 & -91.0 \\ \hline 632.9 & 761.5 & -69.8 & -378.0 & -20.2 & 105.1 \\ \hline 369.4 & 765.9 & 437.3 & 75.5 & -148.2 & -98.4 \\ \hline 135.1 & 363.0 & 348.2 & 372.7 & 217.3 & 74.1 \end{array}\right]$$

$$= \{\underset{V_{mod}^{(1)}}{4122.1} \mid \underset{V_{mod}^{(2)}}{1208.5} \mid \underset{V_{mod}^{(3)}}{444.6} \mid \underset{V_{mod}^{(4)}}{257.9} \mid \underset{V_{mod}^{(5)}}{106.1} \mid \underset{V_{mod}^{(6)}}{29.1}\}$$

Applying the SRSS procedure:

$$\mathbf{V^{srss}} = \sqrt{(4122.1)^2 + (1208.5)^2 + (444.6)^2 + (257.9)^2 + (106.1)^2 + (29.1)^2}$$
$$= 4327.6 \quad \mathbf{kN}$$

Maximum credible overturning moment

The overturning moment for each story and mode is obtained using Eq. (119):

$$\mathbf{M_j^{(i)}} = \sum_{k=j+1}^{n} \left[(\mathbf{h_k} - \mathbf{h_j}) \cdot \mathbf{F_j^{(i)}} \right]$$

Table 8 - Example 7 - Modal story overturning moments

story	$M_{mod}^{(1)}$ (kN · m)	$M_{mod}^{(2)}$ (kN · m)	$M_{mod}^{(3)}$ (kN · m)	$M_{mod}^{(4)}$ (kN · m)	$M_{mod}^{(5)}$ (kN · m)	$M_{mod}^{(6)}$ (kN · m)
6	0.0	0.0	0.0	0.0	0.0	0.0
5	3324.9	-2246.7	1209.8	-679.2	237.8	-55.9
4	9698.6	-5287.8	1828.7	-290.9	-111.0	61.9
3	18652.9	-7333.6	1015.6	272.2	60.2	-93.3
2	29505.8	-7094.7	-6.9	-298.7	170.9	66.8
1	41466.8	-4558.2	282.4	-643.1	-162.9	-68.2
0	53833.1	-932.7	1616.3	130.7	155.3	19.2

Now using the SRSS procedure for example to the fourth story:

$$\mathbf{M}_4^{srss} = \sqrt{(9698.6)^2 + (-5287.7)^2 + (1828.7)^2 + (-290.9)^2 + (-111.0)^2 + (61.9)^2}$$
$$= \mathbf{4080.2 \quad kN}$$

The result, in kN · m, for all stories is:

$$\begin{array}{l} \\ \left\{\mathbf{M}^{srss}\right\} = \end{array} \begin{array}{c} \\ \left\{\begin{array}{r} 0.0 \\ \hline 4252.9 \\ \hline 11201.3 \\ \hline 20070.6 \\ \hline 30348.8 \\ \hline 41722.9 \\ \hline 53865.8 \end{array}\right\} \end{array} \begin{array}{c} \downarrow \textbf{story} \\ 6 \\ 5 \\ 4 \\ 3 \\ 2 \\ 1 \\ 0 \end{array}$$

<u>Maximum credible base overturning moment</u>

Base overturning moment contributed by each mode can be computed from:

$$\left\{\mathbf{M}_{mod}\right\} = \left\{\mathbf{h}\right\}^T \left[\mathbf{F}_{mod}\right] = \left\{18 \mid 15 \mid 12 \mid 9 \mid 6 \mid 3\right\} \left[\begin{array}{r|r|r|r|r|r} 1108.3 & -748.9 & 403.3 & -226.4 & 79.3 & -18.6 \\ \hline 1016.2 & -264.8 & -196.9 & 355.8 & -195.6 & 57.9 \\ \hline 860.2 & 331.8 & -477.4 & 58.2 & 173.4 & -91.0 \\ \hline 632.9 & 761.5 & -69.8 & -378.0 & -20.2 & 105.1 \\ \hline 369.4 & 765.9 & 437.3 & 75.5 & -148.2 & -98.4 \\ \hline 135.1 & 363.0 & 348.2 & 372.7 & 217.3 & 74.1 \end{array}\right]$$

$$= \left\{\begin{array}{c|c|c|c|c|c} 53833 & -933 & 1616 & 131 & 155 & 19 \\ \mathbf{M}_{mod}^{(1)} & \mathbf{M}_{mod}^{(2)} & \mathbf{M}_{mod}^{(3)} & \mathbf{M}_{mod}^{(4)} & \mathbf{M}_{mod}^{(5)} & \mathbf{M}_{mod}^{(6)} \end{array}\right\}$$

and

$$\mathbf{M}^{srss} = \sqrt{(53833.1)^2 + (-932.7)^2 + (1616.3)^2 + (130.7)^2 + (155.3)^2 + (19.2)^2}$$
$$= \mathbf{53865.8 \quad kN \cdot m}$$

<u>Static equivalent lateral forces</u>

These forces, in kN, are computed using Eq. (123) using the story shears obtained by using the SRSS procedure:

$$
\left\{ \mathbf{F^E} \right\} = \left\{ \begin{array}{l} \mathbf{1417.6} \\ \hline \mathbf{2369.8 - 1417.6} \\ \hline \mathbf{3080.3 - 2369.8} \\ \hline \mathbf{3640.1 - 3080.3} \\ \hline \mathbf{4080.2 - 3640.1} \\ \hline \mathbf{4327.6 - 4080.2} \end{array} \right\} = \pm \left\{ \begin{array}{r} \mathbf{1417.6} \\ \hline \mathbf{951.9} \\ \hline \mathbf{710.5} \\ \hline \mathbf{559.8} \\ \hline \mathbf{440.3} \\ \hline \mathbf{247.6} \end{array} \right\} \begin{array}{c} \downarrow \textbf{story} \\ \mathbf{6} \\ \mathbf{5} \\ \mathbf{4} \\ \mathbf{3} \\ \mathbf{2} \\ \mathbf{1} \end{array}
$$

The overturning moment, in $kN \cdot m$, computed for these equivalent lateral loads is:

$$
\mathbf{M^E} = \left\{ \mathbf{h} \right\}^{\mathbf{T}} \left[\mathbf{F^E} \right] = \left\{ \mathbf{18 \mid 15 \mid 12 \mid 9 \mid 6 \mid 3} \right\} \left\{ \begin{array}{r} \mathbf{1417.6} \\ \hline \mathbf{951.9} \\ \hline \mathbf{710.5} \\ \hline \mathbf{559.8} \\ \hline \mathbf{440.3} \\ \hline \mathbf{247.6} \end{array} \right\} = \mathbf{56742.1} \quad \mathbf{kN \cdot m}
$$

The overturning moment, in this case, is slightly larger than the one obtained using the SRSS procedure with the modal spectral overturning moments.

In Example 5 the step-by-step response of the system to the El Centro record was computed, in Example 6 the individual modal spectral responses were computed for the spectrum of the same record — thus permitting the computation of the absolute maximum spectral response —, and in Example 7 the SRSS procedure was applied to the results obtained in Example 6. Now some comparisons can be made between the results of the three examples.

Table 9 - Example 7 - Comparison of the results from Examples 5, 6, and 7

Parameter	Example 5 Step-by-step Analysis	Example 6 Modal spectral Absolute value	Example 7 Modal spectral SRSS
Roof lateral displacement	0.149 m	0.160 m	0.149 m
Base shear	4 360 kN	6 170 kN	4 330 kN
Overturning moment	54 400 $kN \cdot m$	56 700 $kN \cdot m$	53 900 $kN \cdot m$

For this case the match between the step-by step analysis values and the values obtained using the SRSS procedure is reasonable good.

■