A granulation of linguistic information in AHP decision-making problems

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Abstract

To be fully utilized, linguistic information present in decision-making, has to be made operational through information granulation. This study is concerned with information granulation present in the problems of Analytic Hierarchy Process (AHP), which is available in the characterization of a pairwise assessment of alternatives studied in the decision-making problem. The granulation of entries of reciprocal matrices forming the cornerstone of the AHP is formulated as an optimization problem in which an inconsistency index is minimized by a suitable mapping of the linguistic terms on the predetermined scale. Particle Swarm Optimization is used as an optimization framework. Both individual and group decision-making models of AHP are discussed.

1. Introduction

In decision-making problems, we are commonly faced with information provided by humans, which is inherently non-numeric. Partial evaluations, preferences, weights are expressed linguistically. The evident role of fuzzy sets in decision-making and associated important processes such as e.g., consensus building is well documented in the literature, see [1,5,6,12].

While fuzzy sets have raised awareness about the non-numeric nature of information, its importance, a need for its handling and provided a great deal of techniques of processing fuzzy sets, the fundamental issue about a transformation of available pieces of linguistic information into formal constructs of information granules. The resulting information granules are afterwards effectively processed within the computing setting pertinent to the assumed framework of information granulation. The linguistic terms such as high, and medium are in common usage. It is not clear, however, how they have to be translated into the entities, which can be further seamlessly processed using the formalisms of sets, fuzzy sets, rough sets and alike. Likewise, it is not straightforward what optimization criterion can be envisioned when arriving at the formalization of the linguistic terms through information granules.

Given the diversity of decision-making problems and being alerted to the fact that each of their categories could come with some underlying specificity and particular requirements, in this study we concentrate on the Analytic Hierarchy Process (AHP) model, which addresses a large and important category of decision-making schemes, see [13,15,16]. There is a visible abundance literature on the refinements and generalizations of these models as well as various applied studies [2,3,7,8,14].

The pairwise comparisons of alternatives are articulated in terms of linguistic quantifications, say highly preferred, moderately preferred, etc. Each term is associated with some numeric values. It has been identified quite early in the development of the AHP-like models that the single numeric values taken from the 1–9 scale do not necessarily fully reflect the complexity of the non-numeric nature of the pairwise comparisons. The first approach along this line was presented in [9] where the authors admitted triangular fuzzy numbers defined in the scale used in the method, say 1–9. There have been a significant number of pursuits along this line. The granular nature of the pairwise assessments was discussed in the context of a group decision-making where reaching consensus calls for some flexibility of evaluations individual assessments have to be endowed with to facilitate processes of consensus building.

In the study, we focus on two scenarios: decision-making involving a single decision – maker and decision-making realized in presence of several decision-makers (group decision-making problem). The granulation formalism being discussed in the study concerns intervals and fuzzy sets however it applies equally well to any other formal scheme of information granulation, say probabilistic sets, shadowed sets, rough sets. It is worth stressing here that information granulation offers an operational model of the AHP to...
be used in presence of linguistic pairwise comparisons. The PSO framework supporting the formation of information granules helps translate linguistic quantification into meaningful information granules so that the highest consistency of the evaluations is achieved.

The presentation is structured as follows. We start with a brief outline of the AHP decision model (Section 2). Section 3 is concerned with a quantification of linguistic terms present in reciprocal matrices where a granulation of the terms leads to the operational realization of further processing forming preference vectors. The optimization of the granulation process is elaborated on in Section 4. Both set-based and fuzzy set-oriented information granulation mechanisms are studied. Group decision-making in which a collection of reciprocal matrices is considered is covered in Section 5. Conclusions are drawn Section 6. The study comes with an illustration of the presented ideas and the ensuing algorithms.

2. The AHP method – a brief overview

Given a finite number of alternatives, say various options, solutions, etc. $a_1, a_2, \ldots, a_m$ etc. to be considered in a certain investment scenario, the objective is to order them by associating with them some degrees of preference expressed in the $[0, 1]$ interval. The essence of the method introduced by Saaty is to determine such preference values through running a series of pairwise comparisons of the alternatives. The results are organized in an $n \times n$ reciprocal matrix $R = [r_{ij}], \ i, j = 1, 2, \ldots, n$. The matrix exhibits two important features. The diagonal values of the matrix are equal to 1. The entries that are symmetrically positioned with respect to the diagonal satisfy the condition of reciprocality that is $r_{ij} = 1/r_{ji}$.

The starting point of the estimation process of the fuzzy set of preferences is entries of the reciprocal matrix which are obtained through collecting results of pairwise evaluations offered by an expert, designer or user (depending on the character of the task at hand). Prior to making any assessment, the expert is provided with a finite scale with values in-between 1–7. Some other alternatives of the scales such as those involving 5 or 9 levels could be sought as well. If $a_i$ is strongly preferred over $a_j$ when being considered in the context of the fuzzy set whose membership function we would like to estimate, then this judgment is expressed by assigning high values of the available scale, say 6 or 7. If we still sense that $a_j$ is preferred over $a_i$ yet the strength of this preference is lower in comparison with the previous case, then this is quantified using some intermediate values of the scale, say 3 or 4. If no difference is considered, the values close to 1 are the preferred choice, say 2 or 1. The value of 1 indicates that $a_i$ and $a_j$ are equally preferred. The general quantification of preferences positioned on the scale of 1–9 can be described as follows:

- equal importance 1,
- moderate importance of one element over another 3,
- strong importance 5,
- demonstrated importance 7,
- extreme importance 9.

There are also some intermediate values, which could be used to further quantify the relative dominance. On the other hand, if $a_j$ is preferred over $a_i$, the corresponding entry assumes values below one. Given the reciprocal nature of the assessment, once the preference of $a_j$ over $a_i$ has been quantified, the inverse of this number is inserted into the entry of the matrix that is located at the $(j, i)$-th coordinate. Next the maximal eigenvalue is computed along with its corresponding eigenvector. The normalized version of the eigenvector is then the membership function of the fuzzy set we considered when doing all pairwise assessments of the elements of its universe of discourse. The effort to complete pairwise evaluations is far more manageable in comparison to any experimental overhead we need when assigning membership grades to all elements (alternatives) of the universe in a single step. Practically, the pairwise comparison helps the expert focus only on two elements once at a time thus reducing uncertainty and hesitation while leading to the higher level of consistency. The assessments are not free of bias and could exhibit some inconsistency. In particular, we cannot expect that the transitivity requirement could be fully satisfied. Fortunately, the lack of consistency could be quantified and monitored. The largest eigenvalue computed for $R$ is always greater than the dimensionality of the reciprocal matrix (recall that in reciprocal matrices the elements positioned symmetrically along the main diagonal are inverse of each other), $\lambda_{\text{max}} > n$ where the equality $\lambda_{\text{max}} = n$ occurs only if the results are fully consistent. The ratio

$$
\nu = (\lambda_{\text{max}} - n)/(n - 1)
$$

(1)
can be regarded as a certain consistency index of the data; the higher its value, the less consistent are the collected experimental results. This expression can be sought as the indicator of the quality of the pairwise assessments provided by the expert. If the value of $\nu$ is too high exceeding a certain superimposed threshold, the experiment may need to be repeated. Typically if $\nu$ is less than a certain threshold level, the assessment is sought to be consistent while higher values of $\nu$ call for the re-examination of the experimental data and a re-run of the experiment. The threshold of the consistency ratio (CR) expressed as the ratio of the consistency index, CI and the random consistency index (RI), $\text{CR} = \text{CI}/\text{RI}$ is also established (typically assuming the value of 0.1) to assess the quality of the results of pairwise comparison, cf. [4].

3. A quantification (granulation) of linguistic terms as their operational realization

The linguistic terms used in a pairwise comparison of alternatives are expressed linguistically by admitting qualitative terms. They can be organized in a linear fashion, as there is some apparent linear order among them. The terms themselves are not operational meaning that no further processing can be realized, which involves a quantification of the linguistic terms. Schematically, we can portray the process of arriving at the operational representation of linguistic terms as illustrated in Fig. 1. In this figure, capital letters denote the corresponding linguistic terms: $L$ – low, $M$ – medium, $H$ – high.

The two important features of such granulation mechanisms are worth noting here: (a) the mapping is by no means linear that is a localization of the associated information granules on the scale is not uniform, and (b) the semantics of the terms allocated in the process of granulation is retained. Various information granulation formal-
isms can be contemplated including sets (intervals), rough sets, fuzzy sets, and shadowed sets [10], just to mention several alternatives.

The question on how to arrive at the operational version of the information granules can be reformulated as a certain optimization problem. To achieve high flexibility when stating the optimization problem, especially expressing a minimized optimization criterion, we use a technique of Particle Swarm Optimization (PSO) [17–22] as a viable vehicle. This population-based method offers a significant level of diversity of possible objective functions, which play a role of fitness functions.

4. The optimization of granulation problem

The construction of information granules is realized as a certain optimization problem. In this section, we elaborate on the fitness function, its realization, and the PSO optimization along with the corresponding formation of the components of the swarm.

4.1. Evaluation of the mapping from linguistic terms to information granules

The objective of the fitness function is to provide a quantification of the information granules on which information granules are to be mapped. Considering the nature of the AHP model, the quality of the solution (preference vector) is expressed in terms of the inconsistency index. For the given vector of cutoff points, their quality associates with the corresponding value of the inconsistency index. The minimization of the values of the index by adjusting the positions of the cutoff points in the 1–9 scale is realized by the PSO. When it comes to the formation of the fitness function, its determination has to take into account a fact that interval-valued entries of the reciprocal matrix have to return numeric values of the fitness function. This is realized as follows. As we encounter information granules in the form of intervals, we consider a series of their realizations. Let us consider that an individual generated by the PSO has produced a collection of cutoff points in the 1–9 scale. Considering the nature of the AHP model, the quality of the solution (preference vector) is expressed in terms of the inconsistency index. For the given vector of cutoff points, their quality associates with the corresponding value of the inconsistency index. The minimization of the values of the index by adjusting the positions of the cutoff points in the 1–9 scale is realized by the PSO. When it comes to the formation of the fitness function, its determination has to take into account a fact that interval-valued entries of the reciprocal matrix have to return numeric values of the fitness function. This is realized as follows. As we encounter information granules in the form of intervals, we consider a series of their realizations. Let us consider that an individual generated by the PSO has produced a collection of cutoff points in the 1–9 scale.

Example 1. We consider a 5 by 5 reciprocal matrix with the three linguistic entries

\[
R = \begin{bmatrix}
1 & H & L & 1/L & 1/L \\
1/H & 1 & 1/L & 1/M & 1/M \\
1/L & 1 & 1/M & 1/M & 1/M \\
L & M & M & 1 & 1/L \\
L & M & M & L & 1
\end{bmatrix}
\]

The granular matrix \( R \) is sampled 500 time (the numbers drawn from the uniform distribution defined over the corresponding sub-intervals of the [1,9] scale). Recall that the fitness function is the average of the inconsistency index computed over each collection of 500 reciprocal matrices. The process of learning realized by the PSO is illustrated in Fig. 2 where we show the values of the fitness function obtained in successive generations. In the PSO we use the generic form of the algorithm where the updates of the velocity are realized in the form \( \mathbf{v}(t + 1) = \mathbf{v}(t) + c_1 \mathbf{a} \cdot (\mathbf{x}_p - \mathbf{x}) + c_2 \mathbf{b} \cdot (\mathbf{x}_g - \mathbf{x}) \) where “\( t \)” is an index of the generation and - denotes a vector multiplication realized coordinatewise. \( \mathbf{a} \) and \( \mathbf{b} \) are vectors of random numbers generated from a uniform distribution expressed over the unit interval, \( \mathbf{x}_p \) represents the local best solution and \( \mathbf{x}_g \) represents the global best solution. The next position of the particle is computed in a straightforward manner, \( \mathbf{x}(t + 1) = \mathbf{x}(t) + \mathbf{v}(t + 1) \). The constriction coefficient \( \zeta \) decreases linearly over successive generations, \( \zeta(t) = (\text{num}_\text{iter} - t) \cdot (\zeta_{\text{max}} - \zeta_{\text{min}})/\text{num}_\text{iter} + \zeta_{\text{min}} \), where \( \zeta_{\text{max}} = 0.9, \zeta_{\text{min}} = 0.4 \) and “num_iter” is the total number of generations the PSO and “\( t \)” denotes the index of the current generation.

To put the obtained optimization results in a certain context, we report the performance obtained when considering a uniform distribution of the cutoff points over the scale, which are equal to 3.67 and 6.34, respectively. The average inconsistency index assumes

\[
\frac{1}{6} \cdot (\zeta_{\text{max}} + \zeta_{\text{min}}) = 4.91.
\]

The progression of the optimization is quantified in terms of the fitness function obtained in successive generations, see Fig. 2.

Fig. 3 shows the distribution of the values of the inconsistency index for a uniform distribution of the cutoff points.

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the value of 0.1159 with a standard deviation of 0.0332. The histogram of the inconsistency rates in Fig. 3 provides a more comprehensive view at the results: there is a visible presence of a longer tail of the distribution spread towards higher values of the inconsistency index.

The PSO returns the optimal cutoff points of 2.2 and 2.4, which are evidently shifted towards the lower end of the scale. The inconsistency index takes on now lower values and is equal to 0.054 with the standard deviation of 0.0252. The corresponding histogram is shown in Fig. 4.

For the optimal splits of the scale, a reciprocal matrix with the lowest inconsistency index is given below:

\[
R = \begin{bmatrix}
1 & 2.2 & 1/1.65 & 1/2.35 & 1/2.4 \\
1/2.2 & 1 & 1/2.1 & 1/4.05 & 1/6.15 \\
1.65 & 2.1 & 1 & 1/1.45 & 1/2.45 \\
2.35 & 4.05 & 1.45 & 1 & 1/1.25 \\
2.4 & 6.15 & 2.45 & 1.25 & 1
\end{bmatrix}
\]

with the normalized eigenvector corresponding to the largest eigenvalue of this reciprocal matrix equal to \( \mathbf{e} = [0.79 \ 1.00 \ 0.650 \ 3.000]^T \), which identifies the second alternative as an optimal one.

**Example 2.** Here we consider another 5 \( \times \) 5 reciprocal matrix with 5 linguistic terms, VL, L, M, H, and VH

\[
R = \begin{bmatrix}
1 & L & M & VL & H \\
1/L & 1 & M & 1/VL & VH \\
1/M & 1/M & 1 & 1/H & L \\
1/VL & VL & H & 1 & H \\
1/H & 1/VH & 1/L & 1/H & 1
\end{bmatrix}
\]

The results of the optimization are shown in Fig. 5, here the parameters of the PSO were set up as follows: number of particles: 100, number of iterations: 500, \( c_1 = c_2 = 2 \).

The results corresponding with the uniform distribution of the cutoff points (that is 2.6, 4.2, 5.8, and 7.4) come with the average inconsistency index of 0.0888 with a standard deviation of 0.0162. The PSO produces the cutoff points of 1.1, 1.8, 3.3, and 5.6. The inconsistency index is now lower and equal to 0.0205 with the standard deviation of 0.0102. The corresponding histograms both for the uniform and PSO-optimized cutoff points are shown in Fig. 6.

For the optimal split of the scale, a reciprocal matrix with the lowest inconsistency index is given below:

\[
R = \begin{bmatrix}
1 & 1.1 & 4.55 & 1.05 & 5.5 \\
1/1.1 & 1 & 3.85 & 1/1.05 & 6.05 \\
1/4.55 & 1/3.85 & 1 & 1/5.3 & 1.3 \\
1/1.05 & 1.05 & 5.3 & 1 & 5.3 \\
1/5.5 & 1/6.05 & 1/1.3 & 1/5.3 & 1
\end{bmatrix}
\]
with the corresponding eigenvalue being equal to $\mathbf{e} = [1.00 \ 0.92 \ 0.06 \ 0.97 \ 0.00]^T$.

**Example 3.** We consider a $7 \times 7$ reciprocal matrix with 5 linguistic terms as before, that is VL, L, M, H, and VH

\[
R = \begin{bmatrix}
1 & 1/M & 1/VH & 1/H & 1/M & 1/VL & 1/L \\
M & 1 & 1/L & 1/VL & 1/VL & M & \ M \\
VL & L & 1 & H & VL & H & VH \\
H & VL & 1/H & 1/L & L & L & L \\
M & VL & 1/VL & L & 1 & L & H \\
VL & 1/M & 1/H & 1/L & L & L & L \\
L & 1/M & 1/VH & 1/L & 1/H & 1/L & 1 \\
\end{bmatrix}
\]

The parameters of the PSO are the same as in the previous examples. The PSO optimization takes place in the first generations of the method, Fig. 7.

The obtained results are as follows:

Uniform distribution of the cutoff points: $\{2.6, 4.2, 5.8, 7.4\}$. The average inconsistency index assumes the value of 0.1184 with a standard deviation of 0.0153, the histogram of the inconsistency index is included in Fig. 8.

PSO-optimized cutoff points: $\{1.5, 1.7, 2.3, 2.4\}$, average inconsistency index $0.0269 \pm 0.0075$. The corresponding histogram of the values of the inconsistency index is illustrated in Fig. 9. Not only the average value of the inconsistency index is lower but its standard deviation is also reduced to 50% of the one encountered when having a uniform distribution of the cutoff points.

For the optimal splits of the $[1,9]$ interval, a reciprocal matrix with the lowest inconsistency index is given below:

\[
R = \begin{bmatrix}
1 & 1/2.15 & 1/3.9 & 1/2.3 & 1/2.25 & 1/1.25 & 1/1.6 \\
2.15 & 1 & 1/1.65 & 1 & 1/1.05 & 1.8 & 2.2 \\
3.9 & 1.65 & 1 & 2.3 & 1.15 & 2.35 & 3.8 \\
2.3 & 1 & 1/2.3 & 1 & 1/1.5 & 1.65 & 1.55 \\
2.25 & 1.05 & 1/1.15 & 1.5 & 1 & 1.6 & 2.35 \\
1.25 & 1/1.8 & 1/2.35 & 1/1.65 & 1/1.6 & 1 & 1.55 \\
1.6 & 1/2.2 & 1/3.8 & 1/1.55 & 1/2.35 & 1/1.55 & 1 \\
\end{bmatrix}
\]

where the eigenvector associated with the maximal eigenvalue is $\mathbf{e} = [0 \ 0.47 \ 1.00 \ 0.36 \ 0.59 \ 0.16 \ 0.06]^T$. For comparison, the uniform distribution of the cutoff points yields $\mathbf{e} = [1.00 \ 0.63 \ 0.00 \ 0.73 \ 0.43 \ 0.91 \ 0.97]^T$.

### 4.2. Fuzzy sets in the quantification of linguistic terms

The quantification of the linguistic terms with the aid of fuzzy sets can be realized by further refining the representation completed through intervals of the scale. The intent is to form fuzzy sets by using the cutoff points already formed when building the intervals. This helps assess the performance of this model of the linguistic terms versus the one of the interval character.
Consider the intervals already formed over the [1,9] scale with the cutoff points shown in Fig. 10.

Based on them we form triangular fuzzy sets whose modal values are adjustable. In the construct above the fuzzy sets overlap at 0.5 at the cutoff points as shown in Fig. 10.

When selecting randomly the realization of the linguistic terms, they come with the membership values computed on the basis of the membership functions. For instance, if we randomly draw the values \( a_1 \) and \( a_2 \) as the realizations of the linguistic terms L (Low) and H (High) present in the reciprocal matrix the results (viz. the eigenvalue and its associated eigenvector) come with the figure of merit computed based on the aggregation of the membership degrees of L and H that is \( L(a_1) \times H(a_2) \) where “\( \times \)” stands for any t-norm. Note that the values are drawn randomly from the supports of the linguistic terms L and H. Denote it by \( \mu \) where \( \mu = L(a_1) \times H(a_2) \). The calculated inconsistency index \( v \) is now associated with the value of \( \mu \). In a nutshell, we slightly augment the procedure of evaluating the fitness function discussed for the interval realization of linguistic terms. The main difference is that in the calculations of the fitness function used by the PSO to optimize the modal values of the fuzzy sets of the linguistic terms we use the weighted sum of the form

\[
\sum_{k=1}^{5} \mu_k v_k
\]

with the summation completed over all the samples drawn, \( k = 1,2,\ldots,K \). The above formula reduces to the one used in the previous case when the membership functions are replaced by the characteristic functions.

An alternative approach is to start with the construction of fuzzy sets of the linguistic terms. In this case the content of the particle is a collection of the modal values of the triangular membership functions. We assume that the consecutive membership functions intersect at the level of \( \frac{1}{2} \); this implies that the fuzzy sets are completely specified by their modal values.

**Example 4.** We use the same reciprocal matrix as in Example 1. Using interval granulation, the best cutoffs of interval are 2.2 and 2.4. By making use of the fuzzy sets in scale granulation and with the sample of 500 elements and the fitness function expressed by (1) the average fitness function \( Q \) is equal to 0.0533, and other results are from 0.0533 to 0.0635. The PSO optimized fuzzy sets are shown in Fig. 11.

The fuzzy set of preferences comes as \( e = [0.84 \ 1.00 \ 0.830\ 470.00]^T \).

Proceeding with the second approach, viz. constructing fuzzy sets from scratch, the PSO-optimized cutoff points are 1.4, 2.6, and 2.8 for which the performance index \( Q \) is equal to 0.051 (which is very close to the value of the fitness function obtained in the first approach). The progression of the optimization process is illustrated in Fig. 12.

The fuzzy set of preferences has the following entries \( e = [0.81 \ 1.00 \ 0.84 \ 0.37 \ 0.00]^T \). The results are practically the same as obtained when starting with the interval quantification of the linguistic terms.

5. A Group decision-making scenario

The AHP model is used in the realization of group decision-making processes. Here we assume “c” reciprocal matrices. Each decision-maker uses the same number of linguistic terms. The optimization is realized in the same way as before. The main difference is with regard to the performance index which is taken as a sum of the inconsistency indexes for all reciprocal matrices \( R[1], R[2], \ldots, R[c] \). Here we distinguish between two scenarios, namely (a) each reciprocal matrix involves the same number of linguistic terms, and (b) there are different numbers of the linguistic terms. The fitness function to be minimized is the sum of the inconsistency indexes of the corresponding reciprocal matrices, namely

\[
Q = \sum_{i=1}^{c} v[i]
\]

Fig. 12. Q in successive PSO generations.

The joint treatment of the linguistic terms coming from the experts engaged in the process of group decision-making allows us to treat these terms in a unified fashion and reconcile their semantics so that the individual rankings are made comparable and thus could be aggregated to arrive at the joint view at the alternatives. A granulation of the linguistic terms realized at the level of individual decision-makers involved in the group decision-making may result in results of individual rankings that are more difficult to aggregate.

**Example 5.** We consider 4 decision-makers whose assessments of alternatives are presented in the form of the following reciprocal matrices (with 5 linguistic terms); note there is the same number of linguistic terms used in all reciprocal matrices

\[
R_1 = \begin{bmatrix}
1 & L & M & VL & H \\
1/L & 1 & M & 1/VL & H \\
1/M & 1/M & 1 & H & L \\
1/VL & VL & H & 1 & H \\
1/H & 1/VH & 1/L & 1/H & 1
\end{bmatrix}
\]

and

\[
R_2 = \begin{bmatrix}
1 & VL & 1/L & 1/M & 1/VH \\
1/VL & 1 & 1/L & 1/H & 1 \\
1/L & L & 1 & 1/L & 1/VH \\
1/M & VL & H & 1 & H \\
1/H & 1/VH & 1/L & 1/H & 1
\end{bmatrix}
\]
eigenvalues of the reciprocal matrices are as follows:

\[
R_3 = \begin{bmatrix}
1 & H & VH & M & VH \\
1/H & 1 & 1/L & 1/M & VH \\
1/VH & L & 1 & 1/VL & L \\
1/M & M & VL & 1 & H \\
1/VH & 1/VL & 1/L & 1/H & 1
\end{bmatrix}
\]

The corresponding preference vectors associated with the maximal
individual. This means that four separate optimization problems are
matrices and come up with the granulation of the scale done indi-
vidually. This means that four separate optimization problems are
solved. The results are as follows:

\[
R_1 = \begin{bmatrix}
1 & L_1 & M_1 & M_1 & H_1 \\
1/L_1 & 1 & M_1 & H_1 & M_1 \\
1/M_1 & 1/H_1 & L_1 & 1 & L_1 \\
1/H_1 & 1/M_1 & 1/L_1 & 1/L_1 & 1
\end{bmatrix}
\]

The cutoff points obtained here are 1.5, 1.8, 3.8, and 4.0. The
realization of the linguistic terms was unified throughout the
group of experts. In contrast, let us consider the same reciprocal
matrices and come up with the granulation of the scale done indi-
vidually. This means that four separate optimization problems are
solved. The results are as follows:

\[
R_3 = \begin{bmatrix}
1 & M_3 & H_3 & L_3 & M_3 \\
1/M_3 & 1 & M_3 & L_3 & H_3 \\
1/H_3 & 1/M_3 & 1 & L_3 & L_3 \\
1/L_3 & 1/L_3 & M_3 & 1 & H_3 \\
1/M_3 & 1/L_3 & 1/L_3 & M_3 & 1
\end{bmatrix}
\]

The histograms of the inconsistency rates in Figs. 14 and 15 pro-
vide a more comprehensive view at the results of the realization of
the linguistic terms. The minimization process is defined so that the
maximal inconsistency indexes of the reciprocal matrices are dis-
played in Fig. 13.

The minimzed fitness function is now equal to 0.1392 ± 0.049.
The corresponding preference vectors associated with the maximal
eigenvalues of the reciprocal matrices are as follows:

\[
e(1) = [1.00 \ 0.69 \ 0.12 \ 0.97 \ 0.00]^T
\]

\[
e(2) = [1.00 \ 0.99 \ 0.83 \ 0.34 \ 0.00]^T
\]

\[
e(3) = [1.00 \ 0.07 \ 0.15 \ 0.37 \ 0.00]^T
\]

\[
e(4) = [1.00 \ 0.95 \ 0.87 \ 0.00 \ 0.85]^T
\]

The cutoff points obtained here are 1.5, 1.8, 3.8, and 4.0. The
realization of the linguistic terms was unified throughout the
group of experts. In contrast, let us consider the same reciprocal
matrices and come up with the granulation of the scale done indi-
vidually. This means that four separate optimization problems are
solved. The results are as follows:

\[
R_1 = \begin{bmatrix}
1 & L_1 & M_1 & L_1 & H_1 \\
1/L_1 & 1 & M_1 & L_1 & H_1 \\
1/M_1 & 1/H_1 & L_1 & 1 & L_1 \\
1/H_1 & 1/M_1 & 1/L_1 & 1/L_1 & 1
\end{bmatrix}
\]

The cutoff points obtained here are 1.5, 1.8, 3.8, and 4.0. The
realization of the linguistic terms was unified throughout the
group of experts. In contrast, let us consider the same reciprocal
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R_1 = \begin{bmatrix}
1 & L_1 & M_1 & L_1 & H_1 \\
1/L_1 & 1 & M_1 & L_1 & H_1 \\
1/M_1 & 1/H_1 & L_1 & 1 & L_1 \\
1/H_1 & 1/M_1 & 1/L_1 & 1/L_1 & 1
\end{bmatrix}
\]

The cutoff points obtained here are 1.5, 1.8, 3.8, and 4.0. The
realization of the linguistic terms was unified throughout the
group of experts. In contrast, let us consider the same reciprocal
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vidually. This means that four separate optimization problems are
solved. The results are as follows:

\[
R_1 = \begin{bmatrix}
1 & L_1 & M_1 & L_1 & H_1 \\
1/L_1 & 1 & M_1 & L_1 & H_1 \\
1/M_1 & 1/H_1 & L_1 & 1 & L_1 \\
1/H_1 & 1/M_1 & 1/L_1 & 1/L_1 & 1
\end{bmatrix}
\]

The cutoff points obtained here are 1.5, 1.8, 3.8, and 4.0. The
realization of the linguistic terms was unified throughout the
group of experts. In contrast, let us consider the same reciprocal
matrices and come up with the granulation of the scale done indi-
vidually. This means that four separate optimization problems are
solved. The results are as follows:
Fig. 14. The distribution of the inconsistency index in case of a uniform distribution of cutoff points: (a) $R_1$ (b) $R_2$ (c) $R_3$ (d) $R_4$.

Fig. 15. The PSO-optimized distribution of the inconsistency index: (a) $R_1$ (b) $R_2$ (c) $R_3$ (d) $R_4$. 
The cutoff points obtained here are 1.7, 3.3, 3.4, 6.3. If we solve the optimization problem separately for each matrix, the results are:

\[ e[1] = [1.00 \ 0.99 \ 0.98 \ 0.00 \ 0.24]^T \]

\[ e[2] = [1.00 \ 0.98 \ 0.11 \ 0.09 \ 0.00]^T \]

\[ e[3] = [0.00 \ 0.01 \ 0.02 \ 1.00 \ 0.76]^T \]

\[ e[4] = [0.00 \ 0.52 \ 0.56 \ 0.65 \ 1.00]^T \]

The cutoff points obtained are 1.7, 3.3, 3.4, 6.3.

6. Conclusions

The study presented in this paper provided the methodology and the algorithmic framework of constructing preference vectors on a basis of reciprocal matrices with the linguistically quantified results of pairwise comparisons. It is important to underline that while linguistic information is readily available, it is not operational and thus the phase of information granulation becomes indispensable. The mapping of the linguistic assessments to the corresponding information granules makes the linguistic information operational so that the final preference values are determined.

The formation of a suite of information granules realizing a linguistic (symbolic) quantification as a result of a given optimization problem equips these granules with well-articulated semantics.

The use of PSO as an optimization environment offers a great deal of flexibility. Different fitness functions could be easily accommodated. Likewise a multiobjective optimization can be sought. The need for the two-objective optimization becomes apparent in case of a group decision-making where in addition to the criterion of consistency of evaluations applied to the individual reciprocal matrices, one can consider a minimization of dispersions among the vectors of preferences associated with these matrices.

While we have predominantly focused our discussion on the granulation of linguistic terms in the language of intervals and fuzzy sets (to some extent), the discussed methodology applies equally well in case of other formalisms of information granules. In particular, dealing with probabilistically granulated linguistic terms could help shed light on possible linkages between probabilistic and fuzzy models of decision-making along with some possible hybrid probabilistic-fuzzy schemes.

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References